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Using approximate secant equations in limited memory methods for multilevel unconstrained optimization

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Abstract

The properties of multilevel optimization problems defined on a hierarchy of discretization grids can be used to define approximate secant equations, which describe the second-order behaviour of the objective function. Following earlier work by Gratton and Toint (2009), we introduce a quasi-Newton method (with a linesearch) and a nonlinear conjugate gradient method that both take advantage of this new second-order information. We then present numerical experiments with these methods and formulate recommendations for their practical use.

Keywords nonlinear optimization · multilevel problems · quasi-Newton methods · nonlinear conjugate gradient methods · limited-memory algorithms.

Mathematics Subject Classification (2000) 65K05 · 65K10 · 90C06 · 90C26 · 90C30 · 90C53.

1 Introduction

Many optimization problems in science and engineering exhibit a hierarchical structure, especially when they are derived from the discretization of underlying continuous applications on grids of varying size. Inspired in part by multigrid techniques in linear algebra, numerical methods for the efficient resolution of such problems have been considered by various authors (see Fisher, 1998, Nash, 2000, Oh, Milstein, Bouman and Webb, 2003, Gratton, Sartenaer and Toint, 2008*b*, Gratton, Mouffe, Toint and Weber-Mendonça, 2008*a*, for instance). While globally efficient, many of the proposed methods still have difficulties in that estimating the local curvature of the considered objective function might be costly, especially on finer grids where the number of variables is large, even if the associated matrices often have a sparsity pattern reflecting the grid structure or are only considered in operator form. These difficulties have been addressed, for large but otherwise unstructured optimization problems, by iterative methods such as limited-memory quasi-Newton techniques (see for instance, the L-BFGS method of Liu and Nocedal, 1989), in which the Hessian matrix is assembled at each iteration as a product of a modest number of low-rank updates. More recently, Gratton and Toint (2009) have proposed an extension of this approach to problems with multilevel grid structure, where the hierarchy of grids is used to generate additional curvature information. This proposal however includes many variants and leaves the door open as to which of these is preferable in terms of numerical reliability and efficiency. It is the purpose of this paper to explore this issue and make specific algorithmic recommendations.

The paper is organized as follows. A statement of the problem and more detailed review of the considered methods is proposed in Section 2, while algorithmic variants of these method are discussed in Section 3. Our numerical experiments are then presented in Section 4 and conclusions are finally drawn in Section 5. The detailed numerical results are reported in appendix for reference purposes.

2 The problem

We consider the minimization of a smooth nonlinear objective function f from $\mathbb{R}^n$ to $\mathbb{R}$:

$$\min_{x \in \mathbb{R}^n} f(x). \tag{2.1}$$

Quasi-Newton methods solve this problem in an iterative way that involves the construction, around the current iterate $x_k \in \mathbb{R}^n$, of a second-order model of the objective function of the form

$$m_k(x_k + s) = f(x_k) + \langle g_k, s \rangle + \tfrac{1}{2} \langle s, B_k s \rangle ,$$

where $g_k \stackrel{\text{def}}{=} \nabla_x f(x_k)$, and B_k is a symmetric (often positive-definite) approximation of the Hessian matrix $\nabla_{xx} f(x_k)$, capturing the information about the curvature of the objective function around x_k. If positive-definiteness of the matrix B_k is maintained throughout the iterations, the search direction at iteration k is then computed as

$$d_k = -B_k^{-1} g_k, \tag{2.2}$$

and a linesearch is performed along this direction to ensure global convergence (see Dennis and Schnabel, 1983, pages 116-125). In this process, the new approximate Hessian matrix B_{k+1} is typically updated from the previous one, such that the secant equation

$$B_{k+1} s_k = y_k \tag{2.3}$$

holds, where $s_k = x_{k+1} - x_k$ and $y_k = g_{k+1} - g_k$. This condition arises from the mean value theorem for vector-valued functions, which implies that (2.3) is satisfied by the mean Hessian on the interval $[x_k, x_{k+1}]$. The pair (s_k, y_k) is said to be the secant pair associated with equation (2.3). To avoid the cost of solving a linear system in (2.2), the inverse matrix $H_k \stackrel{\text{def}}{=} B_k^{-1}$ is often recurred instead of B_k. The most famous updating process of this kind is the BFGS formula

$$H_{k+1} = \left(I - \frac{s_k y_k^T}{s_k^T y_k} \right) H_k \left(I - \frac{y_k s_k^T}{s_k^T y_k} \right) + \frac{s_k s_k^T}{s_k^T y_k} \tag{2.4}$$

which was developed by Broyden (1970), Fletcher (1970), Goldfarb (1970), and Shanno (1970). It readily follows from this formula that H_{k+1} remains positive definite if H_k is positive definite and

$$s_k^T y_k > 0, \tag{2.5}$$

a condition that one can always enforce in the linesearch procedure if the objective function is bounded below (again see Dennis and Schnabel, 1983, pages 120, 208).

As indicated above, we are especially interested in the resolution of multilevel (unconstrained) optimization problems, that is problems that are defined at several levels of accuracy. We denote these levels with index i from the finest ($i = r$) to the coarsest ($i = 0$) descriptions. Prolongation operators P_i and restriction operators R_i are given to go from level $i-1$ to level i and *vice versa*. At their finest level, such problems often have a large number of variables, making the explicit storage of the (dense) matrices B_k or H_k impractical. Gratton and Toint (2009) introduce a limited-memory quasi-Newton method that takes advantage of the multilevel hierarchy, in that they construct new (approximate) secant pairs

$$(S_i s_k, S_i y_k) \quad \text{for} \quad 0 \le i < r, \tag{2.6}$$

where $S_i \stackrel{\text{def}}{=} P_r \cdots P_{i+1} R_{i+1} \cdots R_r$. These are filtered versions of the secant pair (s_k, y_k) whose oscillatory components were mainly removed, bringing hence to the algorithm a wider range of curvature information on the objective function, and helping it to reduce faster the smooth modes of the error. The numerical results show a significant decrease of the number of iterations and function evaluations when these smoothed pairs are used.

Nonlinear conjugate gradient methods constitute another set of iterative methods that can tackle large-scale problems with small memory requirements, and were introduced by Fletcher and Reeves (1964). In these methods, the search direction d_{k+1} is updated from the previous one as

$$d_{k+1} = -M_k g_{k+1} + \beta_k d_k \tag{2.7}$$

where we may use a symmetric positive definite preconditioner M_k, and where β_k is some conjugacy factor (see Hager and Zhang, 2006*b*, for a survey). A linesearch is then performed in this direction. A standard strategy to choose the preconditioner M_k is to take an approximation to $\nabla_{xx} f(x_*)^{-1}$ coming from a quasi-Newton formula. In particular, we may set M_k to our last approximation H_{k+1} of the inverse Hessian, and use the smoothed pairs given in (2.6) to define a limited-memory BFGS preconditioner for the nonlinear conjugate gradient method.

However, the combination of all these potentially useful strategies leads to a substantial number of algorithmic variants, whose numerical efficiency and robustness have not yet been analyzed. It is the purpose of this paper to present systematic numerical experiments involving these variants, that is the various ways in which smoothed secant pairs may be used inside either a quasi-Newton or a nonlinear conjugate gradient method.

3 Algorithmic choices

In order to make our experiments as systematic as possible, we now detail several aspects of the considered algorithmic variants, namely the choice of the conjugacy factor used to define the search direction (Section 3.1), the choice of the linesearch algorithm (Section 3.2), and the computation of the approximate inverse Hessian using the smoothed pairs (Section 3.3).

3.1 Conjugacy factor

We consider three choices for the conjugacy factor β_k. The first one is

$$\beta_k^{QN} = 0,$$

corresponding to the quasi-Newton methods, since the search direction given by (2.7) is then the same as in (2.2) with $M_k = H_{k+1}$. The next two choices are advised by Hager and Zhang (2005) as the best choices actually known. Hence, our second choice is

$$\beta_k^{HZ} = \left(M_k y_k - 2 d_k \frac{y_k^T M_k y_k}{d_k^T y_k} \right)^T \frac{g_{k+1}}{d_k^T y_k},$$

which was proposed in the same paper, and our third choice is an hybrid of the formulae of Hestenes and Stiefel (1952) and Dai and Yuan (1999):

$$\beta_k^{DYHS} = \max \left[0, \min \left[\frac{y_k^T M_k g_{k+1}}{d_k^T y_k}, \frac{g_k^T M_k g_{k+1}}{d_k^T y_k} \right] \right],$$

which was introduced by Dai and Yuan (2001).

3.2 Linesearch

We selected two linesearch algorithms for our tests. The first one was described by Dennis and Schnabel (1983), and used by Gratton and Toint (2009) for their numerical experiments. It determines a step length $\alpha_k > 0$ that ensures the satisfaction of the Wolfe conditions:

$$f(x_k + \alpha_k d_k) \leq \delta \alpha_k g_k^T d_k, \tag{3.8}$$

$$g_{k+1}^T d_k \geq \sigma g_k^T d_k. \tag{3.9}$$

for some parameters $0 < \delta \leq \sigma < 1$. We translated the matlab code of Gratton and Toint (2009) to Fortran 95, and kept the default parameters $\delta = 10^{-4}$ and $\sigma = 0.9$. The second linesearch algorithm comes from the CG_DESCENT code (version 3.0) of Hager and Zhang (2006*a*), which ensures the satisfaction either of the Wolfe conditions (3.8)–(3.9) or of the approximate Wolfe conditions:

$$\sigma g_k^T d_k \leq g_{k+1}^T d_k \leq (2\delta - 1) g_k^T d_k,$$

where $0 < \delta \leq \sigma < 1$. We chose the variant of their algorithm that uses first the Wolfe conditions and switch permanently to the approximate ones as soon as the function variation is relatively small, that is when $|f(x_{k+1}) - f(x_k)| \leq \omega C_k$, where

$$\begin{cases} Q_k = 1 + Q_{k-1}\Delta, & Q_{-1} = 0, \\ C_k = C_{k-1} + (|f(x_k)| - C_{k-1})/Q_k, & C_{-1} = 0, \end{cases}$$

for some parameters Δ and ω in the interval $[0, 1]$. The practical implementation is a Fortran 95 translation of the CG_DESCENT code written in C, and uses the default parameters $\delta = 0.1$, $\sigma = 0.9$, $\Delta = 0.7$ and $\omega = 10^{-3}$.

3.3 L-BFGS update

We assume that at most m pairs may be stored. So we consider, at each iteration, a set $\mathcal{P}_k$ containing $m_k \leq m$ pairs $(s_{k,j}, y_{k,j})$, for $j = 1, \ldots, m_k$, that provide the available curvature information on the objective function. The approximate inverse Hessian H_{k+1} is then defined using the L-BFGS formula, that is $H_{k+1} = H_{k,m_k}$ where

$$H_{k,j} = \left(I - \rho_{k,j} s_{k,j} y_{k,j}^T\right) H_{k,j-1} \left(I - \rho_{k,j} y_{k,j} s_{k,j}^T\right) + \rho_{k,j} s_{k,j} s_{k,j}^T$$

with $\rho_{k,j} = (s_{k,j}^T y_{k,j})^{-1}$, for $j = 1, \ldots, m_k$, and some chosen initial matrix $H_{k,0}$. As mentioned by many authors, the choice of this initialization matrix $H_{k,0}$ is quite important to get a well scaled search direction. We thus use the classical initialization

$$H_{k,0} = \gamma_k I \quad \text{with} \quad \gamma_k = \frac{s_k^T y_k}{y_k^T y_k}.$$

By the way, since we are only interested in the products $H_{k+1} g_{k+1}$, we need not to explicitly construct the matrix H_{k+1}, but may use the (matrix-free) L-BFGS two-loop recursion instead (see Nocedal, 1980).

The set $\mathcal{P}_k$ may contain *exact* pairs of the form (s_k, y_k) and *smoothed* pairs of the form $(S_i s_k, S_i y_k)$. Different variants were proposed by Gratton and Toint (2009) to select the pairs kept in memory:

- the `L-BFGS` strategy only uses the m last exact secant pairs;
- the `Full` strategy uses the m last generated pairs, making no difference between smoothed and exact pairs;
- the `Local` strategy uses only the $\min(r, m-1)$ smoothed pairs from the current iteration, in addition to past and current exact pairs;
- the `Mless` (*memory less*) strategy uses only (smoothed and exact) pairs from the current iteration.

Each exact pair is always integrated in the update after its smoothed versions. However, we still may choice the order of the smoothed pairs in the L-BFGS formula:

- the `Coarse first` strategy first integrates the pairs that have been smoothed on the coarser levels (that is integrating pairs $(S_i s_k, S_i y_k)$ with index i running from 0 to $r - 1$);
- the `Fine first` strategy first integrates the pairs that have been smoothed on the finer levels (that is integrating pairs $(S_i s_k, S_i y_k)$ with index i running from $r - 1$ to 0).

Name	Level	Size	Type	Description
DNT	8	511	1-D, quadratic	Dirichlet to Neumann transfer
P2D	8	261121	2-D, quadratic	Poisson model problem
P3D	5	250047	3-D, quadratic	Poisson model problem
DEPT	8	261121	2-D, quadratic	Elastic-plastic torsion problem
DODC	8	261121	2-D, convex	Optimal design with composite materials
MINS-SB	8	261121	2-D, convex	Minimal surface problem
MINS-OB	8	261121	2-D, convex	Minimal surface problem
MINS-DMSA	8	261121	2-D, convex	Minimal surface problem
IGNISC	8	261121	2-D, convex	Combustion problem
DSSC	8	261121	2-D, convex	Combustion problem
BRATU	8	261121	2-D, convex	Combustion problem
NCCS	7	130050	2-D, nonconvex	Optimal control problem
NCCO	7	130050	2-D, nonconvex	Optimal control problem
MOREBV	8	261121	2-D, nonconvex	Boundary value problem

Table 1: Test problems set

Gratton and Toint (2009) moreover propose to control the collinearity of the smoothed pairs with respect to the original exact pair; the pairs that do not satisfy the condition

$$|\langle S_i s_k, s\rangle| \leq \tau \|S_i s_k\|_2 \|s_k\|_2, \tag{3.10}$$

for some $\tau \in (0,1]$, are thus discarded. Additionally, we enforce the positivity constraint (2.5) by ignoring pairs that do not satisfy

$$|\langle S_i s_k, S_i y_k\rangle| \leq \mu \langle s_k, y_k\rangle,$$

for parameter $\mu \in (0,1)$. We test the values 0.999 and 1.0 for the threshold τ, and set μ to 10^{-6}.

4 Numerical experiments

We now present numerical experiments whose objective is to clarify which of the above algorithmic options provides the most reliable and efficient method for solving grid-structured unconstrained optimization problems. All codes used in these experiments are written in Fortran 95, and the runs were performed on a bi-processor Intel Xeon X5482 (4 cores, 3.20 GHz) with 64 GB of RAM.

4.1 Test problems

In our tests, we consider the unconstrained optimization problems from the set provided by Gratton, Mouffe, Sartenaer, Toint and Tomanos (2009). A small description of each problem is given in Table 1 with the level and size at which it was considered.

4.2 Starting point and stopping criterion

We choose the starting point as $[x_0]_j = 0.5$ and consider that the algorithm has converged as soon as $\|g_k\|_\infty \leq 10^{-5}$.

4.3 Results

We ran a large number (78) of possible combinations of the variants described in Section 3 on our set of 14 test problems and report all results of the 1092 runs on comet-shape graphs representing a measure of the effort spent in function/gradient evaluations vs. iterations number. More precisely, we have first scaled, separately for each test problem, on the one hand, the number of function evaluations plus five times the number of gradient evaluations, and on the other hand, the iterations number, by dividing them by the best obtained for this problem by all algorithmic variants. We then plotted the averages of these scaled measures on all test problems for each algorithmic variant separately, after removing the variants which fail on at least one problem (the number of such variants is given in legend). All variants use $m = 9$, but experiments with other choices (not reported here) give similar results.

In the first of these plots (Figure 1), we have used squares for the variants where the quasi-Newton search direction (2.2) is chosen, stars for the variants where the conjugate-gradient search direction (2.7) with the conjugacy factor β_k^{DYHS} is chosen, and triangles for the variants where the conjugate-gradient search direction (2.7) with the conjugacy factor β_k^{HZ} is chosen.

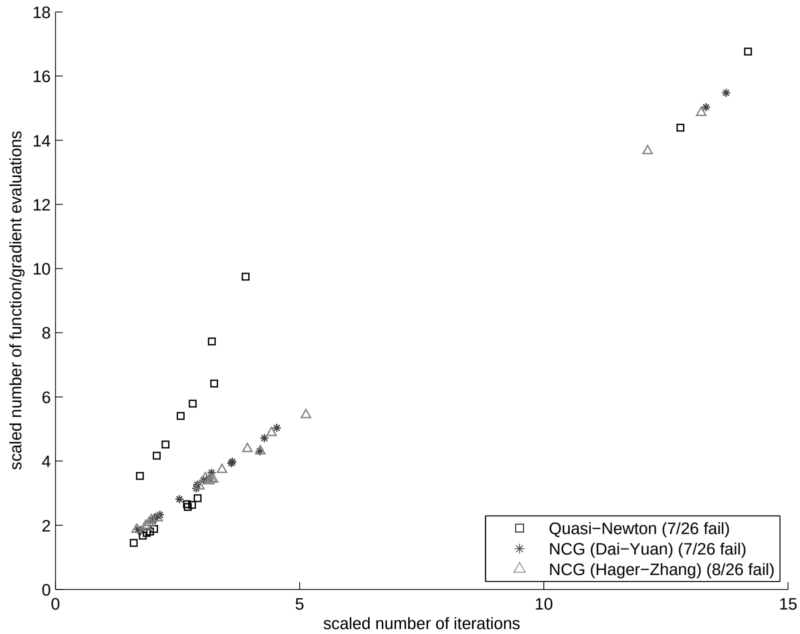

Figure 1: Comet-shape graph comparing the choice of conjugacy factor

We note a substantial spread of the results, with some options being up to 15 times worse than others. The worst cases (in the top right corner) correspond to combination of the Hager-Zhang linesearch, the `Mless` strategy and the collinearity threshold set to 0.999. Among the other variants, we may also observe a second set of variants which requires (in average) twice the number of function/gradient evaluations. These correspond to the use of the Hager-Zhang linesearch inside the quasi-Newton method. The third set of variants gathered results from the quasi-Newton method (with the Dennis-Schnabel linesearch) and nonlinear conjugate gradient method, with in average, better results for the former methods than for the latter.

We next compare the effect of the linesearch choice in Figure 2. In that picture, squares have been used for the variants using the Hager-Zhang linesearch, and stars for the variants using the Dennis-Schnabel linesearch.

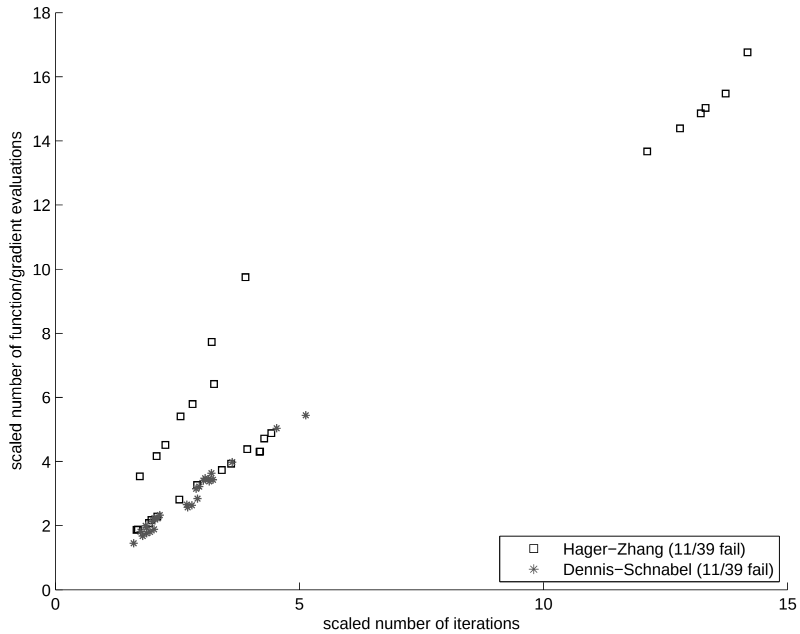

Figure 2: Comet-shape graph comparing the choice of linesearch

We observe that the Hager-Zhang linesearch do not improve the performance, especially inside the quasi-Newton method, where the number of function and gradient evaluations per iterations is doubled in average. We would therefore advise the Dennis-Schnabel linesearch since it is also simpler.

Figure 3 compares the variants performance on the basis of the pairs selection strategy. This time, squares have been used for the variants using the `L-BFGS` strategy, stars for the `Full` strategy, triangles for the `Local` strategy, and crosses for the `Mless` strategy.

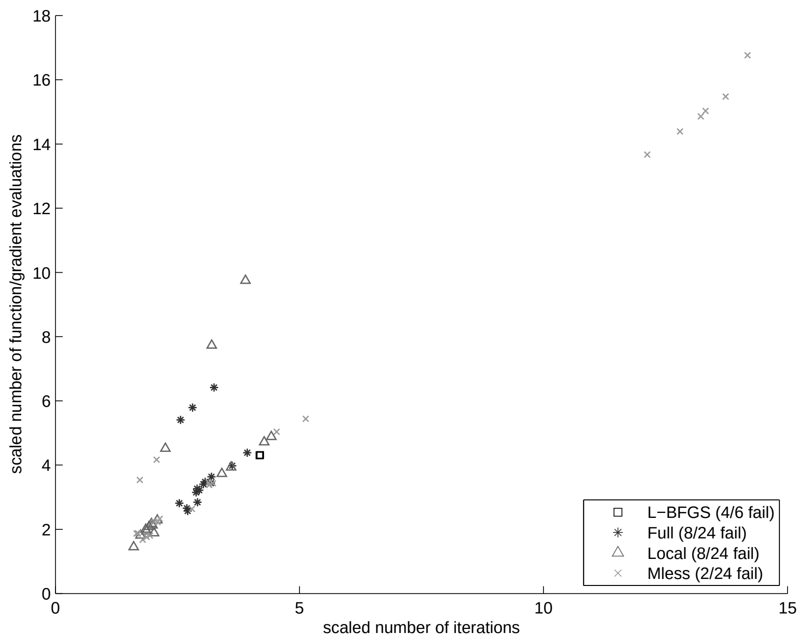

Figure 3: Comet-shape graph comparing the choice of pairs selection

We observe that the `L-BFGS` strategies are in a factor 4, in average, from the best ones. The `Local` and `Mless` strategies give best results. We recommend the `Local` strategy since the number of pairs used by the `Mless` strategy is limited by the number of levels, and thus prevent the full use of the available memory capacity.

We then compare the effect of the integration order of the smoothed pairs inside the L-BFGS update. In Figure 4, we have thus used squares for the variants that integrate first the pairs smoothed at the coarser levels (`Coarse First` strategy), and stars for the variants that integrate first the pairs smoothed at the finer levels (`Fine First` strategy).

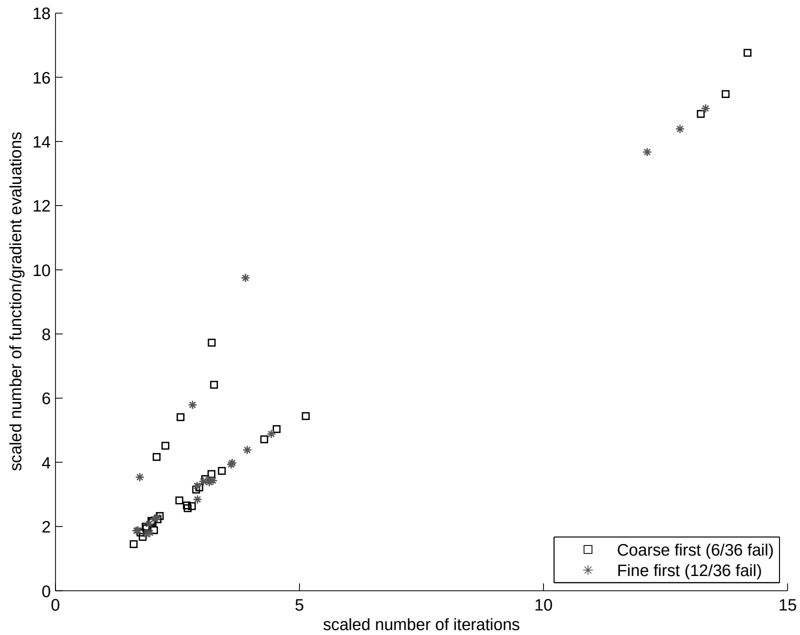

Figure 4: Comet-shape graph comparing the choice of pairs order

It is unclear which strategy is the more efficient. Nevertheless, the variants using the `Coarse First` strategy encounter less failures.

Finally, we consider the interest to control the collinearity of the smoothed pairs with respect to the exact secant pair from which they were generated. In Figure 5, we have thus used squares to represent the variants that do not control this collinearity (setting the threshold τ to 1.0), and stars to represent the variants that control this collinearity, with a threshold τ set to 0.999.

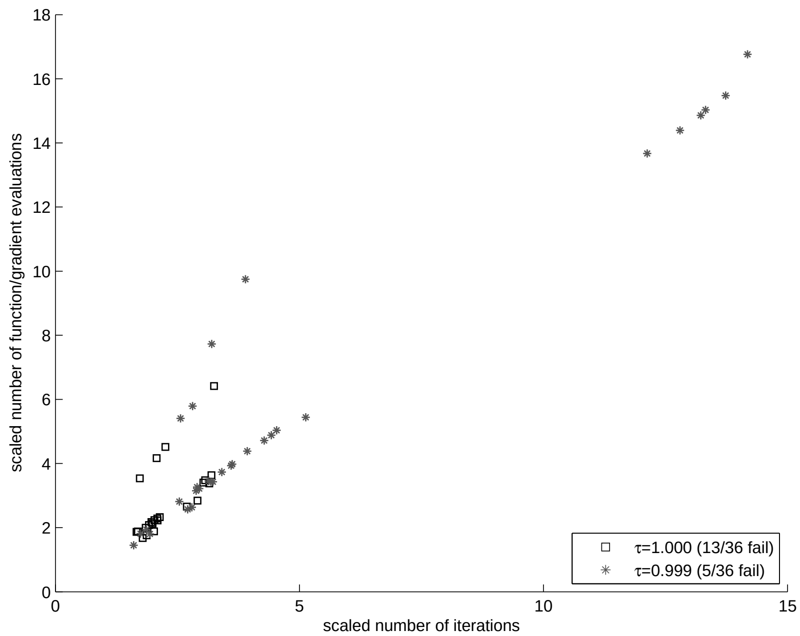

Figure 5: Comet-shape graph comparing the choice of pairs collinearity control

The more efficient variant is again unclear, but the collinearity control seems to improve the robustness.

To summarize our conclusions so far, we may first attempt to distinguish the best variant *not* using smoothed secant pairs, and then compare the strategies using the smoothed pairs with this selected contender, and again look for the best of the set. These two steps are illustrated by Figures 6 and 7, where we have restricted ourselves to considering performance in terms of function and gradient evaluations (the figures for the number of iterations are similar). Both figures are performance profiles (see Dolan and Moré, 2002), a now standard technique to present such results. In these profiles, the proportion of test problems solved by each variant using a number of functions/gradients evaluation within a factor σ of the best performance is plotted against σ for $\sigma \geq 1$. Thus the algorithmic variant whose curve is on top is considered best.

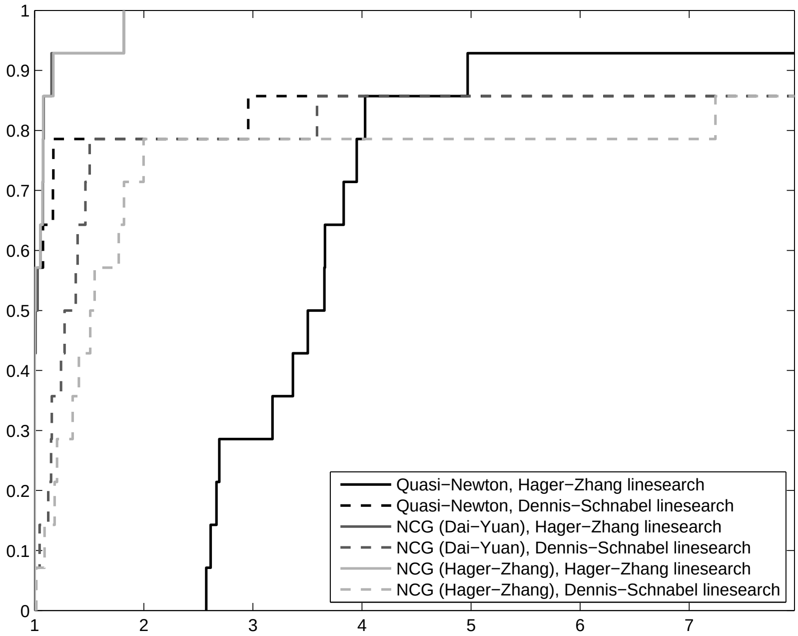

Figure 6: Performance profile for the variants *not* using smoothed secant pairs (proportion of test problems as a function of σ)

The first of these figures illustrates our findings well: the best method in our tests appears to be that using either the Dai-Yuan or the Hager-Zhang formula for deriving the search direction, coupled with the Hager-Zhang linesearch. Interestingly, if one restricts one's attention to quasi-Newton methods ($\beta_k = 0$), then the Dennis-Schnabel linesearch seems to dominate Hager-Zhang's by a substantial margin on our examples.

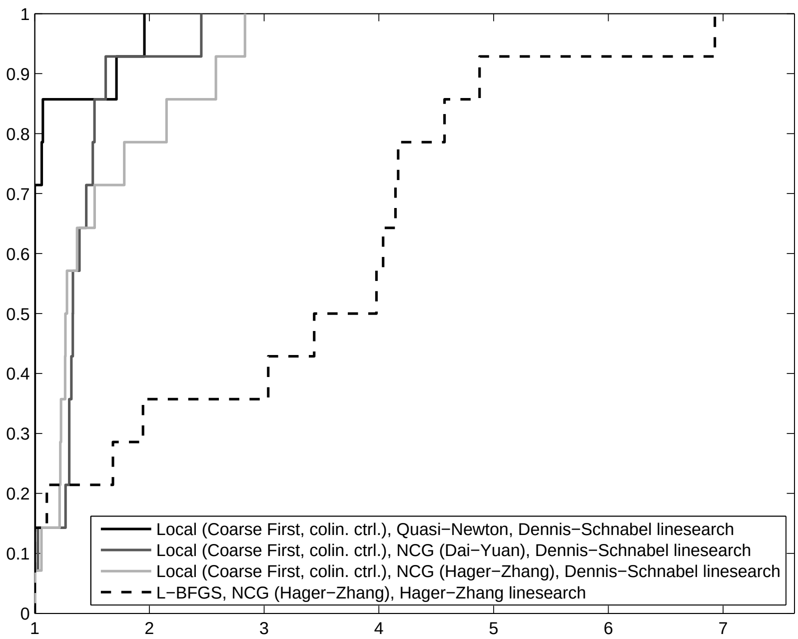

Figure 7: Performance profile for the variants using local smoothed secant pairs against the best one not using them

The second figure also stresses our earlier conclusions: the use of smoothed secant pairs (in their "local" flavour, which we already selected as best above) is clearly beneficial when possible. Indeed, all variants using them substantially dominate the best variant which does not. Amongst them the best variant is the quasi-Newton method using local pairs, starting from the coarsest, together with collinearity control and Dennis-Schnabel linesearch. The gap between this variant and the second best (the CG variant using the Dai-Yuan formula) is significant, although not as wide as that separating the variants using smoothed secant pairs from their best contender.

5 Conclusions

We have considered several variants of the multilevel limited-memory quasi-Newton and conjugate-gradients algorithms for unconstrained optimization and have conducted numerical tests on a battery of examples arising from optimization in the context of partial differential optimzation and involving a hierarchy of grid-based discretizations. Based on these tests, we have singled out a specific limited-memory variant (using the Dennis-Schnabel linesearch technique, local secant pairs applied starting from the coarsest grid level and collinearity control) as the most efficient and reliable algorithm amongst those tested. Our examples also show further systematic interplay between linesearch techniques and search direction formulae whose detailed interpretation remains unclear at this stage.

While we believe the presented conclusions supporting the use of smoothed secant pairs are valuable, we also realize that the full potential of multilevel limited-memory optimization method can only be asserted by continuous use in a wider range of applications than that on which the current tests were performed. Further work in this direction is therefore very desirable. The range of concerned scientific and engineering fields is wide, ranging from optimal control of systems governed by partial differential equations to data assimilation in weather prediction and oceanography, and interesting applications in these areas is expected in the future.

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A Detailed numerical results

In Tables 2 to 15, we display the details of our numerical results (each problem having its own table of results). The first column indicates the chosen conjugacy factor β (see Section 3.1): either the quasi-Newton (QN) choice, or the Dai-Yuan-Hestenes-Stiefel choice (DYHS), or the Hager-Zhang choice (HZ). The second column indicates the linesearch algorithm that was used (see Section 3.2): either the Hager-Zhang one (HZ), or the Dennis-Schnabel one (DS). The third and fourth columns indicate the strategies used to select and order, respectively, the secant pairs (see Section 3.3), while the fifth column gives the value of parameter τ in equation (3.10). The sixth column indicates the execution return status:

- 0: the algorithm ran successfully;
- 4: the Hager-Zhang linesearch failed because of too many secant steps;
- 8: the Hager-Zhang linesearch failed;
- 12: the Dennis-Schnabel linesearch failed;
- -7: the function value became `-Inf`.

The last three columns indicates the number of iterations (nit), function evaluations (nf), and gradient evaluations (ng), respectively.

β	LS	select	order	τ	status	nit	nf	ng
QN	HZ	LBFGS	—	1.000	4	512	1107	1619
QN	HZ	Full	Coarse	1.000	0	259	355	614
QN	HZ	Full	Coarse	0.999	0	179	257	436
QN	HZ	Full	Fine	1.000	0	206	282	488
QN	HZ	Full	Fine	0.999	0	168	225	393
QN	HZ	Local	Coarse	1.000	0	257	368	625
QN	HZ	Local	Coarse	0.999	0	273	403	676
QN	HZ	Local	Fine	1.000	0	225	332	557
QN	HZ	Local	Fine	0.999	0	224	351	575
QN	HZ	Mless	Coarse	1.000	0	388	551	939
QN	HZ	Mless	Coarse	0.999	0	344	474	818
QN	HZ	Mless	Fine	1.000	0	167	220	387
QN	HZ	Mless	Fine	0.999	0	225	307	532
QN	DS	LBFGS	—	1.000	0	635	647	636
QN	DS	Full	Coarse	1.000	0	239	297	240
QN	DS	Full	Coarse	0.999	0	215	261	216
QN	DS	Full	Fine	1.000	0	188	232	189
QN	DS	Full	Fine	0.999	0	148	177	149
QN	DS	Local	Coarse	1.000	0	258	313	259
QN	DS	Local	Coarse	0.999	0	188	225	189
QN	DS	Local	Fine	1.000	0	218	272	219
QN	DS	Local	Fine	0.999	0	222	265	223
QN	DS	Mless	Coarse	1.000	0	364	460	366
QN	DS	Mless	Coarse	0.999	0	357	440	359
QN	DS	Mless	Fine	1.000	0	211	276	212
QN	DS	Mless	Fine	0.999	0	194	248	195
DYHS	HZ	LBFGS	—	1.000	0	184	369	185
DYHS	HZ	Full	Coarse	1.000	0	158	317	160
DYHS	HZ	Full	Coarse	0.999	0	162	325	165
DYHS	HZ	Full	Fine	1.000	0	162	325	164
DYHS	HZ	Full	Fine	0.999	0	125	251	126
DYHS	HZ	Local	Coarse	1.000	0	243	487	249
DYHS	HZ	Local	Coarse	0.999	0	192	385	198
DYHS	HZ	Local	Fine	1.000	0	203	407	210
DYHS	HZ	Local	Fine	0.999	0	171	343	179
DYHS	HZ	Mless	Coarse	1.000	0	487	975	500
DYHS	HZ	Mless	Coarse	0.999	0	200	401	214
DYHS	HZ	Mless	Fine	1.000	0	183	367	203
DYHS	HZ	Mless	Fine	0.999	0	150	301	169
DYHS	DS	LBFGS	—	1.000	0	662	1328	663
DYHS	DS	Full	Coarse	1.000	0	236	527	237
DYHS	DS	Full	Coarse	0.999	0	198	445	199
DYHS	DS	Full	Fine	1.000	0	199	446	200
DYHS	DS	Full	Fine	0.999	0	135	298	136
DYHS	DS	Local	Coarse	1.000	0	231	507	232
DYHS	DS	Local	Coarse	0.999	0	165	372	166
DYHS	DS	Local	Fine	1.000	0	215	470	216
DYHS	DS	Local	Fine	0.999	0	159	352	160
DYHS	DS	Mless	Coarse	1.000	0	420	952	423
DYHS	DS	Mless	Coarse	0.999	0	358	798	363
DYHS	DS	Mless	Fine	1.000	0	172	398	173
DYHS	DS	Mless	Fine	0.999	0	200	455	203
HZ	HZ	LBFGS	—	1.000	0	184	369	185
HZ	HZ	Full	Coarse	1.000	0	194	389	198
HZ	HZ	Full	Coarse	0.999	0	145	291	146
HZ	HZ	Full	Fine	1.000	0	180	361	184
HZ	HZ	Full	Fine	0.999	0	125	251	126
HZ	HZ	Local	Coarse	1.000	0	181	363	185
HZ	HZ	Local	Coarse	0.999	0	216	433	221
HZ	HZ	Local	Fine	1.000	0	206	413	214
HZ	HZ	Local	Fine	0.999	0	163	327	167
HZ	HZ	Mless	Coarse	1.000	0	371	743	381
HZ	HZ	Mless	Coarse	0.999	0	271	543	284
HZ	HZ	Mless	Fine	1.000	0	152	305	174
HZ	HZ	Mless	Fine	0.999	0	177	355	201
HZ	DS	LBFGS	—	1.000	0	1329	2694	1335
HZ	DS	Full	Coarse	1.000	0	237	534	238
HZ	DS	Full	Coarse	0.999	0	194	432	195
HZ	DS	Full	Fine	1.000	0	191	426	192
HZ	DS	Full	Fine	0.999	0	489	1010	490
HZ	DS	Local	Coarse	1.000	0	229	518	230
HZ	DS	Local	Coarse	0.999	0	221	492	222
HZ	DS	Local	Fine	1.000	0	224	493	225
HZ	DS	Local	Fine	0.999	0	190	413	191
HZ	DS	Mless	Coarse	1.000	0	248	546	249
HZ	DS	Mless	Coarse	0.999	0	290	653	291
HZ	DS	Mless	Fine	1.000	0	253	571	255
HZ	DS	Mless	Fine	0.999	0	369	814	372

Table 2: Results for problem DN (level 8, 261121 variables)

β	LS	select	order	τ	status	nit	nf	ng
QN	HZ	LBFGS	—	1.000	0	992	1983	2975
QN	HZ	Full	Coarse	1.000	0	275	427	702
QN	HZ	Full	Coarse	0.999	0	249	381	630
QN	HZ	Full	Fine	1.000	0	249	407	656
QN	HZ	Full	Fine	0.999	0	228	344	572
QN	HZ	Local	Coarse	1.000	0	318	496	814
QN	HZ	Local	Coarse	0.999	0	252	401	653
QN	HZ	Local	Fine	1.000	0	240	383	623
QN	HZ	Local	Fine	0.999	0	257	421	678
QN	HZ	Mless	Coarse	1.000	0	282	423	705
QN	HZ	Mless	Coarse	0.999	0	189	276	465
QN	HZ	Mless	Fine	1.000	0	154	232	386
QN	HZ	Mless	Fine	0.999	0	171	258	429
QN	DS	LBFGS	—	1.000	0	729	748	730
QN	DS	Full	Coarse	1.000	0	234	287	235
QN	DS	Full	Coarse	0.999	0	197	233	198
QN	DS	Full	Fine	1.000	0	221	263	222
QN	DS	Full	Fine	0.999	0	254	306	255
QN	DS	Local	Coarse	1.000	0	175	222	176
QN	DS	Local	Coarse	0.999	0	221	269	222
QN	DS	Local	Fine	1.000	0	235	283	236
QN	DS	Local	Fine	0.999	0	258	316	259
QN	DS	Mless	Coarse	1.000	0	202	264	204
QN	DS	Mless	Coarse	0.999	0	195	257	198
QN	DS	Mless	Fine	1.000	0	179	231	180
QN	DS	Mless	Fine	0.999	0	211	281	216
DYHS	HZ	LBFGS	—	1.000	0	676	1353	677
DYHS	HZ	Full	Coarse	1.000	0	330	661	341
DYHS	HZ	Full	Coarse	0.999	0	207	415	211
DYHS	HZ	Full	Fine	1.000	0	187	375	190
DYHS	HZ	Full	Fine	0.999	0	190	381	193
DYHS	HZ	Local	Coarse	1.000	0	294	589	301
DYHS	HZ	Local	Coarse	0.999	0	246	493	254
DYHS	HZ	Local	Fine	1.000	0	207	415	213
DYHS	HZ	Local	Fine	0.999	0	245	491	252
DYHS	HZ	Mless	Coarse	1.000	0	226	453	236
DYHS	HZ	Mless	Coarse	0.999	0	250	501	267
DYHS	HZ	Mless	Fine	1.000	0	137	275	151
DYHS	HZ	Mless	Fine	0.999	0	213	427	246
DYHS	DS	LBFGS	—	1.000	0	944	1890	945
DYHS	DS	Full	Coarse	1.000	0	280	611	281
DYHS	DS	Full	Coarse	0.999	0	318	705	320
DYHS	DS	Full	Fine	1.000	0	245	537	246
DYHS	DS	Full	Fine	0.999	0	242	533	243
DYHS	DS	Local	Coarse	1.000	0	294	657	295
DYHS	DS	Local	Coarse	0.999	0	252	554	253
DYHS	DS	Local	Fine	1.000	0	208	455	209
DYHS	DS	Local	Fine	0.999	0	190	413	191
DYHS	DS	Mless	Coarse	1.000	0	283	648	284
DYHS	DS	Mless	Coarse	0.999	0	255	589	257
DYHS	DS	Mless	Fine	1.000	0	202	460	204
DYHS	DS	Mless	Fine	0.999	0	180	412	181
HZ	HZ	LBFGS	—	1.000	0	676	1353	677
HZ	HZ	Full	Coarse	1.000	0	247	495	253
HZ	HZ	Full	Coarse	0.999	0	222	445	232
HZ	HZ	Full	Fine	1.000	0	201	403	203
HZ	HZ	Full	Fine	0.999	0	208	417	211
HZ	HZ	Local	Coarse	1.000	0	209	419	212
HZ	HZ	Local	Coarse	0.999	0	245	491	255
HZ	HZ	Local	Fine	1.000	0	227	455	234
HZ	HZ	Local	Fine	0.999	0	213	427	218
HZ	HZ	Mless	Coarse	1.000	0	211	423	224
HZ	HZ	Mless	Coarse	0.999	0	210	421	229
HZ	HZ	Mless	Fine	1.000	0	163	327	188
HZ	HZ	Mless	Fine	0.999	0	171	343	203
HZ	DS	LBFGS	—	1.000	0	875	1796	877
HZ	DS	Full	Coarse	1.000	0	239	528	240
HZ	DS	Full	Coarse	0.999	0	263	584	264
HZ	DS	Full	Fine	1.000	0	231	523	232
HZ	DS	Full	Fine	0.999	0	228	496	229
HZ	DS	Local	Coarse	1.000	0	277	632	278
HZ	DS	Local	Coarse	0.999	0	551	1142	553
HZ	DS	Local	Fine	1.000	0	222	494	223
HZ	DS	Local	Fine	0.999	0	524	1103	529
HZ	DS	Mless	Coarse	1.000	0	247	554	248
HZ	DS	Mless	Coarse	0.999	0	235	532	237
HZ	DS	Mless	Fine	1.000	0	321	703	322
HZ	DS	Mless	Fine	0.999	0	851	1754	857

Table 3: Results for problem P2D (level 8, 261121 variables)

β	LS	select	order	τ	status	nit	nf	ng
QN	HZ	LBFGS	—	1.000	0	250	501	751
QN	HZ	Full	Coarse	1.000	0	125	185	310
QN	HZ	Full	Coarse	0.999	0	119	174	293
QN	HZ	Full	Fine	1.000	0	114	155	269
QN	HZ	Full	Fine	0.999	0	99	144	243
QN	HZ	Local	Coarse	1.000	0	97	151	248
QN	HZ	Local	Coarse	0.999	0	93	146	239
QN	HZ	Local	Fine	1.000	0	92	150	242
QN	HZ	Local	Fine	0.999	0	100	163	263
QN	HZ	Mless	Coarse	1.000	0	135	191	326
QN	HZ	Mless	Coarse	0.999	0	101	147	248
QN	HZ	Mless	Fine	1.000	0	82	120	202
QN	HZ	Mless	Fine	0.999	0	104	155	259
QN	DS	LBFGS	—	1.000	0	207	212	209
QN	DS	Full	Coarse	1.000	0	114	149	116
QN	DS	Full	Coarse	0.999	0	100	126	102
QN	DS	Full	Fine	1.000	0	91	116	93
QN	DS	Full	Fine	0.999	0	102	135	104
QN	DS	Local	Coarse	1.000	0	96	113	98
QN	DS	Local	Coarse	0.999	0	87	109	89
QN	DS	Local	Fine	1.000	0	98	111	100
QN	DS	Local	Fine	0.999	0	94	105	96
QN	DS	Mless	Coarse	1.000	0	99	127	101
QN	DS	Mless	Coarse	0.999	0	93	126	95
QN	DS	Mless	Fine	1.000	0	107	135	109
QN	DS	Mless	Fine	0.999	0	96	120	98
DYHS	HZ	LBFGS	—	1.000	0	153	307	154
DYHS	HZ	Full	Coarse	1.000	0	107	215	109
DYHS	HZ	Full	Coarse	0.999	0	109	219	113
DYHS	HZ	Full	Fine	1.000	0	92	185	95
DYHS	HZ	Full	Fine	0.999	0	87	175	90
DYHS	HZ	Local	Coarse	1.000	0	95	191	98
DYHS	HZ	Local	Coarse	0.999	0	98	197	100
DYHS	HZ	Local	Fine	1.000	0	100	201	103
DYHS	HZ	Local	Fine	0.999	0	99	199	101
DYHS	HZ	Mless	Coarse	1.000	0	131	263	135
DYHS	HZ	Mless	Coarse	0.999	0	91	183	95
DYHS	HZ	Mless	Fine	1.000	0	107	215	114
DYHS	HZ	Mless	Fine	0.999	0	98	197	103

β	LS	select	order	τ	status	nit	nf	ng
DYHS	DS	LBFGS	—	1.000	0	210	423	212
DYHS	DS	Full	Coarse	1.000	0	108	244	110
DYHS	DS	Full	Coarse	0.999	0	124	273	126
DYHS	DS	Full	Fine	1.000	0	105	248	107
DYHS	DS	Full	Fine	0.999	0	103	235	105
DYHS	DS	Local	Coarse	1.000	0	104	216	106
DYHS	DS	Local	Coarse	0.999	0	100	211	102
DYHS	DS	Local	Fine	1.000	0	90	196	92
DYHS	DS	Local	Fine	0.999	0	82	178	84
DYHS	DS	Mless	Coarse	1.000	0	123	276	125
DYHS	DS	Mless	Coarse	0.999	0	149	337	152
DYHS	DS	Mless	Fine	1.000	0	110	252	112
DYHS	DS	Mless	Fine	0.999	0	112	260	114
HZ	HZ	LBFGS	—	1.000	0	153	307	154
HZ	HZ	Full	Coarse	1.000	0	113	227	118
HZ	HZ	Full	Coarse	0.999	0	103	207	105
HZ	HZ	Full	Fine	1.000	0	112	225	115
HZ	HZ	Full	Fine	0.999	0	92	185	94
HZ	HZ	Local	Coarse	1.000	0	91	183	93
HZ	HZ	Local	Coarse	0.999	0	96	193	98
HZ	HZ	Local	Fine	1.000	0	80	161	83
HZ	HZ	Local	Fine	0.999	0	79	159	82
HZ	HZ	Mless	Coarse	1.000	0	133	267	139
HZ	HZ	Mless	Coarse	0.999	0	110	221	113
HZ	HZ	Mless	Fine	1.000	0	102	205	109
HZ	HZ	Mless	Fine	0.999	0	106	213	111
HZ	DS	LBFGS	—	1.000	0	268	557	270
HZ	DS	Full	Coarse	1.000	0	98	232	100
HZ	DS	Full	Coarse	0.999	0	96	214	98
HZ	DS	Full	Fine	1.000	0	117	265	119
HZ	DS	Full	Fine	0.999	0	91	205	93
HZ	DS	Local	Coarse	1.000	0	98	214	100
HZ	DS	Local	Coarse	0.999	0	93	206	95
HZ	DS	Local	Fine	1.000	0	138	290	140
HZ	DS	Local	Fine	0.999	0	156	320	158
HZ	DS	Mless	Coarse	1.000	0	117	258	119
HZ	DS	Mless	Coarse	0.999	0	160	365	162
HZ	DS	Mless	Fine	1.000	0	203	441	205
HZ	DS	Mless	Fine	0.999	0	174	386	176

Table 4: Results for problem P3D (level 5, 250047 variables)

β	LS	select	order	τ	status	nit	nf	ng
QN	HZ	LBFGS	—	1.000	0	1131	2244	3375
QN	HZ	Full	Coarse	1.000	0	297	460	757
QN	HZ	Full	Coarse	0.999	0	302	467	769
QN	HZ	Full	Fine	1.000	0	214	338	552
QN	HZ	Full	Fine	0.999	0	208	321	529
QN	HZ	Local	Coarse	1.000	0	296	448	744
QN	HZ	Local	Coarse	0.999	0	304	468	772
QN	HZ	Local	Fine	1.000	0	272	433	705
QN	HZ	Local	Fine	0.999	0	317	509	826
QN	HZ	Mless	Coarse	1.000	0	243	360	603
QN	HZ	Mless	Coarse	0.999	0	135	189	324
QN	HZ	Mless	Fine	1.000	0	137	200	337
QN	HZ	Mless	Fine	0.999	0	144	212	356
QN	DS	LBFGS	—	1.000	0	921	945	922
QN	DS	Full	Coarse	1.000	0	219	280	220
QN	DS	Full	Coarse	0.999	0	274	333	275
QN	DS	Full	Fine	1.000	0	255	309	256
QN	DS	Full	Fine	0.999	0	255	309	256
QN	DS	Local	Coarse	1.000	0	255	317	256
QN	DS	Local	Coarse	0.999	0	266	324	267
QN	DS	Local	Fine	1.000	0	241	292	242
QN	DS	Local	Fine	0.999	0	272	336	273
QN	DS	Mless	Coarse	1.000	0	216	277	217
QN	DS	Mless	Coarse	0.999	0	171	216	172
QN	DS	Mless	Fine	1.000	0	169	222	170
QN	DS	Mless	Fine	0.999	0	181	236	185
DYHS	HZ	LBFGS	—	1.000	0	677	1355	678
DYHS	HZ	Full	Coarse	1.000	0	277	555	283
DYHS	HZ	Full	Coarse	0.999	0	241	483	245
DYHS	HZ	Full	Fine	1.000	0	251	503	254
DYHS	HZ	Full	Fine	0.999	0	241	483	248
DYHS	HZ	Local	Coarse	1.000	0	306	613	312
DYHS	HZ	Local	Coarse	0.999	0	230	461	242
DYHS	HZ	Local	Fine	1.000	0	246	493	253
DYHS	HZ	Local	Fine	0.999	0	270	541	277
DYHS	HZ	Mless	Coarse	1.000	0	207	415	222
DYHS	HZ	Mless	Coarse	0.999	0	135	271	146
DYHS	HZ	Mless	Fine	1.000	0	149	299	166
DYHS	HZ	Mless	Fine	0.999	0	202	405	222

β	LS	select	order	τ	status	nit	nf	ng
DYHS	DS	LBFGS	—	1.000	0	840	1682	841
DYHS	DS	Full	Coarse	1.000	0	298	658	299
DYHS	DS	Full	Coarse	0.999	0	253	566	254
DYHS	DS	Full	Fine	1.000	0	246	529	247
DYHS	DS	Full	Fine	0.999	0	226	500	227
DYHS	DS	Local	Coarse	1.000	0	241	531	242
DYHS	DS	Local	Coarse	0.999	0	331	718	332
DYHS	DS	Local	Fine	1.000	0	301	681	303
DYHS	DS	Local	Fine	0.999	0	301	660	302
DYHS	DS	Mless	Coarse	1.000	0	235	544	238
DYHS	DS	Mless	Coarse	0.999	0	191	450	198
DYHS	DS	Mless	Fine	1.000	0	193	439	194
DYHS	DS	Mless	Fine	0.999	0	167	370	168
HZ	HZ	LBFGS	—	1.000	0	677	1355	678
HZ	HZ	Full	Coarse	1.000	0	253	507	257
HZ	HZ	Full	Coarse	0.999	0	236	473	238
HZ	HZ	Full	Fine	1.000	0	217	435	219
HZ	HZ	Full	Fine	0.999	0	215	431	217
HZ	HZ	Local	Coarse	1.000	0	241	483	244
HZ	HZ	Local	Coarse	0.999	0	266	533	277
HZ	HZ	Local	Fine	1.000	0	260	521	266
HZ	HZ	Local	Fine	0.999	0	226	453	238
HZ	HZ	Mless	Coarse	1.000	0	190	381	200
HZ	HZ	Mless	Coarse	0.999	0	261	523	284
HZ	HZ	Mless	Fine	1.000	0	113	227	131
HZ	HZ	Mless	Fine	0.999	0	265	531	294
HZ	DS	LBFGS	—	1.000	0	1220	2519	1222
HZ	DS	Full	Coarse	1.000	0	309	685	310
HZ	DS	Full	Coarse	0.999	0	302	688	303
HZ	DS	Full	Fine	1.000	0	222	504	223
HZ	DS	Full	Fine	0.999	0	371	795	372
HZ	DS	Local	Coarse	1.000	0	189	434	191
HZ	DS	Local	Coarse	0.999	0	216	478	217
HZ	DS	Local	Fine	1.000	0	353	760	354
HZ	DS	Local	Fine	0.999	0	323	693	326
HZ	DS	Mless	Coarse	1.000	0	334	731	335
HZ	DS	Mless	Coarse	0.999	0	280	623	281
HZ	DS	Mless	Fine	1.000	0	508	1077	510
HZ	DS	Mless	Fine	0.999	0	269	597	274

Table 5: Results for problem DEPT (level 8, 261121 variables)

β	LS	select	order	τ	status	nit	nf	ng
QN	HZ	LBFGS	—	1.000	0	1699	3395	5094
QN	HZ	Full	Coarse	1.000	0	400	611	1011
QN	HZ	Full	Coarse	0.999	0	396	609	1005
QN	HZ	Full	Fine	1.000	0	554	864	1418
QN	HZ	Full	Fine	0.999	0	415	653	1068
QN	HZ	Local	Coarse	1.000	0	451	695	1146
QN	HZ	Local	Coarse	0.999	0	613	947	1560
QN	HZ	Local	Fine	1.000	0	383	609	992
QN	HZ	Local	Fine	0.999	0	520	826	1346
QN	HZ	Mless	Coarse	1.000	0	347	527	874
QN	HZ	Mless	Coarse	0.999	0	321	489	810
QN	HZ	Mless	Fine	1.000	0	393	604	997
QN	HZ	Mless	Fine	0.999	0	249	379	628
QN	DS	LBFGS	—	1.000	0	1840	1842	1841
QN	DS	Full	Coarse	1.000	0	423	506	425
QN	DS	Full	Coarse	0.999	0	694	809	695
QN	DS	Full	Fine	1.000	0	400	475	401
QN	DS	Full	Fine	0.999	0	612	713	613
QN	DS	Local	Coarse	1.000	0	602	724	603
QN	DS	Local	Coarse	0.999	0	457	548	458
QN	DS	Local	Fine	1.000	0	669	765	670
QN	DS	Local	Fine	0.999	0	510	598	511
QN	DS	Mless	Coarse	1.000	0	295	378	296
QN	DS	Mless	Coarse	0.999	0	510	607	511
QN	DS	Mless	Fine	1.000	0	475	596	476
QN	DS	Mless	Fine	0.999	0	409	492	410
DYHS	HZ	LBFGS	—	1.000	0	1591	3183	1592
DYHS	HZ	Full	Coarse	1.000	0	477	955	485
DYHS	HZ	Full	Coarse	0.999	0	408	817	415
DYHS	HZ	Full	Fine	1.000	0	287	575	291
DYHS	HZ	Full	Fine	0.999	0	668	1337	681
DYHS	HZ	Local	Coarse	1.000	0	359	719	370
DYHS	HZ	Local	Coarse	0.999	0	425	855	441
DYHS	HZ	Local	Fine	1.000	0	448	897	457
DYHS	HZ	Local	Fine	0.999	0	401	803	408
DYHS	HZ	Mless	Coarse	1.000	0	478	963	502
DYHS	HZ	Mless	Coarse	0.999	0	305	613	329
DYHS	HZ	Mless	Fine	1.000	0	353	710	386
DYHS	HZ	Mless	Fine	0.999	0	388	779	417

β	LS	select	order	τ	status	nit	nf	ng
DYHS	DS	LBFGS	—	1.000	0	1647	3296	1648
DYHS	DS	Full	Coarse	1.000	0	528	1185	529
DYHS	DS	Full	Coarse	0.999	0	671	1462	672
DYHS	DS	Full	Fine	1.000	0	501	1105	502
DYHS	DS	Full	Fine	0.999	0	933	2011	934
DYHS	DS	Local	Coarse	1.000	0	436	960	437
DYHS	DS	Local	Coarse	0.999	0	570	1258	571
DYHS	DS	Local	Fine	1.000	0	738	1603	739
DYHS	DS	Local	Fine	0.999	0	637	1387	638
DYHS	DS	Mless	Coarse	1.000	0	432	994	436
DYHS	DS	Mless	Coarse	0.999	0	464	1047	465
DYHS	DS	Mless	Fine	1.000	0	417	973	418
DYHS	DS	Mless	Fine	0.999	0	526	1149	527
HZ	HZ	LBFGS	—	1.000	0	1689	3379	1690
HZ	HZ	Full	Coarse	1.000	0	377	755	383
HZ	HZ	Full	Coarse	0.999	0	376	753	380
HZ	HZ	Full	Fine	1.000	0	396	793	403
HZ	HZ	Full	Fine	0.999	0	672	1345	681
HZ	HZ	Local	Coarse	1.000	0	458	917	465
HZ	HZ	Local	Coarse	0.999	0	275	551	282
HZ	HZ	Local	Fine	1.000	0	513	1027	525
HZ	HZ	Local	Fine	0.999	0	303	607	309
HZ	HZ	Mless	Coarse	1.000	0	269	543	284
HZ	HZ	Mless	Coarse	0.999	0	222	445	235
HZ	HZ	Mless	Fine	1.000	0	310	621	330
HZ	HZ	Mless	Fine	0.999	0	245	491	268
HZ	DS	LBFGS	—	1.000	0	2434	4934	2435
HZ	DS	Full	Coarse	1.000	0	426	943	427
HZ	DS	Full	Coarse	0.999	0	586	1269	587
HZ	DS	Full	Fine	1.000	0	656	1368	660
HZ	DS	Full	Fine	0.999	0	515	1104	517
HZ	DS	Local	Coarse	1.000	0	308	695	309
HZ	DS	Local	Coarse	0.999	0	500	1091	501
HZ	DS	Local	Fine	1.000	0	875	1811	878
HZ	DS	Local	Fine	0.999	0	526	1126	528
HZ	DS	Mless	Coarse	1.000	0	502	1115	503
HZ	DS	Mless	Coarse	0.999	0	356	776	358
HZ	DS	Mless	Fine	1.000	0	622	1335	629
HZ	DS	Mless	Fine	0.999	0	238	525	240

Table 6: Results for problem DODC (level 8, 261121 variables)

β	LS	select	order	τ	status	nit	nf	ng
QN	HZ	LBFGS	—	1.000	0	1208	2391	3599
QN	HZ	Full	Coarse	1.000	0	823	1304	2127
QN	HZ	Full	Coarse	0.999	0	997	1594	2591
QN	HZ	Full	Fine	1.000	0	800	1312	2112
QN	HZ	Full	Fine	0.999	0	933	1557	2490
QN	HZ	Local	Coarse	1.000	0	771	1219	1990
QN	HZ	Local	Coarse	0.999	0	704	1099	1803
QN	HZ	Local	Fine	1.000	0	806	1344	2150
QN	HZ	Local	Fine	0.999	0	753	1252	2005
QN	HZ	Mless	Coarse	1.000	0	789	1231	2020
QN	HZ	Mless	Coarse	0.999	0	752	1209	1961
QN	HZ	Mless	Fine	1.000	0	665	1089	1754
QN	HZ	Mless	Fine	0.999	0	690	1144	1834
QN	DS	LBFGS	—	1.000	0	1260	1275	1261
QN	DS	Full	Coarse	1.000	0	799	1032	800
QN	DS	Full	Coarse	0.999	0	907	1068	908
QN	DS	Full	Fine	1.000	0	789	926	790
QN	DS	Full	Fine	0.999	0	909	1064	910
QN	DS	Local	Coarse	1.000	0	824	988	825
QN	DS	Local	Coarse	0.999	0	723	883	724
QN	DS	Local	Fine	1.000	0	819	946	820
QN	DS	Local	Fine	0.999	0	816	966	817
QN	DS	Mless	Coarse	1.000	0	812	1037	813
QN	DS	Mless	Coarse	0.999	0	805	1036	806
QN	DS	Mless	Fine	1.000	0	796	1016	797
QN	DS	Mless	Fine	0.999	0	802	989	808
DYHS	HZ	LBFGS	—	1.000	0	1241	2490	1249
DYHS	HZ	Full	Coarse	1.000	0	744	1512	788
DYHS	HZ	Full	Coarse	0.999	0	957	1926	987
DYHS	HZ	Full	Fine	1.000	0	809	1630	836
DYHS	HZ	Full	Fine	0.999	0	953	1910	963
DYHS	HZ	Local	Coarse	1.000	0	966	1949	1005
DYHS	HZ	Local	Coarse	0.999	0	800	1630	854
DYHS	HZ	Local	Fine	1.000	0	501	1007	516
DYHS	HZ	Local	Fine	0.999	0	594	1195	618
DYHS	HZ	Mless	Coarse	1.000	0	749	1534	819
DYHS	HZ	Mless	Coarse	0.999	0	731	1509	826
DYHS	HZ	Mless	Fine	1.000	0	654	1326	692
DYHS	HZ	Mless	Fine	0.999	0	748	1529	839

β	LS	select	order	τ	status	nit	nf	ng
DYHS	DS	LBFGS	—	1.000	0	1251	2504	1252
DYHS	DS	Full	Coarse	1.000	0	854	1929	855
DYHS	DS	Full	Coarse	0.999	0	814	1773	815
DYHS	DS	Full	Fine	1.000	0	992	2154	993
DYHS	DS	Full	Fine	0.999	0	1101	2378	1102
DYHS	DS	Local	Coarse	1.000	0	775	1690	776
DYHS	DS	Local	Coarse	0.999	0	793	1745	794
DYHS	DS	Local	Fine	1.000	0	745	1627	746
DYHS	DS	Local	Fine	0.999	0	831	1817	832
DYHS	DS	Mless	Coarse	1.000	0	776	1828	779
DYHS	DS	Mless	Coarse	0.999	0	859	1979	861
DYHS	DS	Mless	Fine	1.000	0	821	1886	823
DYHS	DS	Mless	Fine	0.999	0	818	1840	824
HZ	HZ	LBFGS	—	1.000	0	1081	2163	1082
HZ	HZ	Full	Coarse	1.000	0	826	1668	859
HZ	HZ	Full	Coarse	0.999	0	951	1910	975
HZ	HZ	Full	Fine	1.000	0	658	1321	675
HZ	HZ	Full	Fine	0.999	0	791	1588	811
HZ	HZ	Local	Coarse	1.000	0	652	1315	674
HZ	HZ	Local	Coarse	0.999	0	552	1122	590
HZ	HZ	Local	Fine	1.000	0	620	1248	636
HZ	HZ	Local	Fine	0.999	0	642	1293	662
HZ	HZ	Mless	Coarse	1.000	0	782	1579	818
HZ	HZ	Mless	Coarse	0.999	0	831	1697	907
HZ	HZ	Mless	Fine	1.000	0	638	1289	673
HZ	HZ	Mless	Fine	0.999	0	850	1746	965
HZ	DS	LBFGS	—	1.000	0	1292	2653	1295
HZ	DS	Full	Coarse	1.000	0	836	1893	837
HZ	DS	Full	Coarse	0.999	0	949	2078	950
HZ	DS	Full	Fine	1.000	0	888	1939	889
HZ	DS	Full	Fine	0.999	0	1033	2290	1035
HZ	DS	Local	Coarse	1.000	0	710	1613	711
HZ	DS	Local	Coarse	0.999	0	764	1684	765
HZ	DS	Local	Fine	1.000	0	896	1950	903
HZ	DS	Local	Fine	0.999	0	866	1900	869
HZ	DS	Mless	Coarse	1.000	0	1087	2454	1088
HZ	DS	Mless	Coarse	0.999	0	1225	2683	1232
HZ	DS	Mless	Fine	1.000	0	1060	2341	1064
HZ	DS	Mless	Fine	0.999	0	1214	2659	1236

Table 7: Results for problem MINS-SB (level 8, 261121 variables)

β	LS	select	order	τ	status	nit	nf	ng
QN	HZ	LBFGS	—	1.000	0	1991	3976	5967
QN	HZ	Full	Coarse	1.000	0	3932	6494	10426
QN	HZ	Full	Coarse	0.999	0	3773	6225	9998
QN	HZ	Full	Fine	1.000	0	4383	7804	12187
QN	HZ	Full	Fine	0.999	0	4565	8063	12628
QN	HZ	Local	Coarse	1.000	0	4758	7981	12739
QN	HZ	Local	Coarse	0.999	0	4324	7199	11523
QN	HZ	Local	Fine	1.000	0	3273	5880	9153
QN	HZ	Local	Fine	0.999	0	3334	5929	9263
QN	HZ	Mless	Coarse	1.000	0	4179	6903	11082
QN	HZ	Mless	Coarse	0.999	0	4200	6909	11109
QN	HZ	Mless	Fine	1.000	0	3683	6352	10035
QN	HZ	Mless	Fine	0.999	0	3758	6391	10149
QN	DS	LBFGS	—	1.000	0	2107	2143	2108
QN	DS	Full	Coarse	1.000	0	3392	4005	3393
QN	DS	Full	Coarse	0.999	0	3917	4596	3918
QN	DS	Full	Fine	1.000	0	4151	4688	4152
QN	DS	Full	Fine	0.999	0	4628	5253	4629
QN	DS	Local	Coarse	1.000	0	4254	4911	4255
QN	DS	Local	Coarse	0.999	0	4342	4987	4344
QN	DS	Local	Fine	1.000	0	3834	4382	3835
QN	DS	Local	Fine	0.999	0	3874	4405	3875
QN	DS	Mless	Coarse	1.000	0	4476	5527	4482
QN	DS	Mless	Coarse	0.999	0	4771	5880	4794
QN	DS	Mless	Fine	1.000	0	4337	5156	4339
QN	DS	Mless	Fine	0.999	0	4352	5204	4368
DYHS	HZ	LBFGS	—	1.000	0	1960	3921	1961
DYHS	HZ	Full	Coarse	1.000	0	3724	7461	3779
DYHS	HZ	Full	Coarse	0.999	0	4028	8069	4085
DYHS	HZ	Full	Fine	1.000	0	4872	9756	4919
DYHS	HZ	Full	Fine	0.999	0	4691	9394	4740
DYHS	HZ	Local	Coarse	1.000	0	4860	9725	4923
DYHS	HZ	Local	Coarse	0.999	0	4472	8957	4535
DYHS	HZ	Local	Fine	1.000	0	3386	6779	3415
DYHS	HZ	Local	Fine	0.999	0	3135	6279	3168
DYHS	HZ	Mless	Coarse	1.000	0	4252	8532	4348
DYHS	HZ	Mless	Coarse	0.999	0	4334	8698	4481
DYHS	HZ	Mless	Fine	1.000	0	3581	7188	3671
DYHS	HZ	Mless	Fine	0.999	0	3443	6965	3740

β	LS	select	order	τ	status	nit	nf	ng
DYHS	DS	LBFGS	—	1.000	0	2083	4168	2084
DYHS	DS	Full	Coarse	1.000	0	3510	7680	3511
DYHS	DS	Full	Coarse	0.999	0	4176	9012	4177
DYHS	DS	Full	Fine	1.000	0	4881	10276	4882
DYHS	DS	Full	Fine	0.999	0	4883	10388	4884
DYHS	DS	Local	Coarse	1.000	0	5107	10875	5108
DYHS	DS	Local	Coarse	0.999	0	4678	10086	4681
DYHS	DS	Local	Fine	1.000	0	4097	8698	4100
DYHS	DS	Local	Fine	0.999	0	4341	9223	4342
DYHS	DS	Mless	Coarse	1.000	0	4931	11171	4951
DYHS	DS	Mless	Coarse	0.999	0	4547	10137	4565
DYHS	DS	Mless	Fine	1.000	0	4229	9306	4232
DYHS	DS	Mless	Fine	0.999	0	4151	9180	4162
HZ	HZ	LBFGS	—	1.000	0	1950	3901	1951
HZ	HZ	Full	Coarse	1.000	0	3917	7852	3973
HZ	HZ	Full	Coarse	0.999	0	4086	8182	4131
HZ	HZ	Full	Fine	1.000	0	4913	9841	4961
HZ	HZ	Full	Fine	0.999	0	4367	8741	4409
HZ	HZ	Local	Coarse	1.000	0	5092	10201	5165
HZ	HZ	Local	Coarse	0.999	0	4475	8966	4536
HZ	HZ	Local	Fine	1.000	0	3287	6580	3307
HZ	HZ	Local	Fine	0.999	0	3366	6739	3398
HZ	HZ	Mless	Coarse	1.000	0	4217	8459	4313
HZ	HZ	Mless	Coarse	0.999	0	4414	8875	4627
HZ	HZ	Mless	Fine	1.000	0	3392	6823	3503
HZ	HZ	Mless	Fine	0.999	0	3461	6972	3660
HZ	DS	LBFGS	—	1.000	0	2712	5563	2715
HZ	DS	Full	Coarse	1.000	0	4143	8995	4144
HZ	DS	Full	Coarse	0.999	0	4268	9141	4272
HZ	DS	Full	Fine	1.000	0	4438	9504	4440
HZ	DS	Full	Fine	0.999	0	4465	9554	4471
HZ	DS	Local	Coarse	1.000	0	4094	8813	4095
HZ	DS	Local	Coarse	0.999	0	4920	10604	4923
HZ	DS	Local	Fine	1.000	0	4969	10625	4989
HZ	DS	Local	Fine	0.999	0	5113	10890	5131
HZ	DS	Mless	Coarse	1.000	0	5973	13180	5979
HZ	DS	Mless	Coarse	0.999	0	7114	15568	7162
HZ	DS	Mless	Fine	1.000	0	6880	14889	6908
HZ	DS	Mless	Fine	0.999	0	7820	16836	7905

Table 8: Results for problem MINS-OB (level 8, 261121 variables)

β	LS	select	order	τ	status	nit	nf	ng
QN	HZ	LBFGS	—	1.000	0	1499	2990	4489
QN	HZ	Full	Coarse	1.000	0	1720	2747	4467
QN	HZ	Full	Coarse	0.999	0	1702	2730	4432
QN	HZ	Full	Fine	1.000	0	1624	2686	4310
QN	HZ	Full	Fine	0.999	0	1781	3020	4801
QN	HZ	Local	Coarse	1.000	0	1700	2727	4427
QN	HZ	Local	Coarse	0.999	0	1609	2581	4190
QN	HZ	Local	Fine	1.000	0	1210	2046	3256
QN	HZ	Local	Fine	0.999	0	1307	2224	3531
QN	HZ	Mless	Coarse	1.000	0	1706	2728	4434
QN	HZ	Mless	Coarse	0.999	0	1580	2543	4123
QN	HZ	Mless	Fine	1.000	0	1401	2310	3711
QN	HZ	Mless	Fine	0.999	0	1633	2760	4393
QN	DS	LBFGS	—	1.000	0	1329	1350	1330
QN	DS	Full	Coarse	1.000	0	1313	1613	1314
QN	DS	Full	Coarse	0.999	0	1660	1950	1661
QN	DS	Full	Fine	1.000	0	1601	1933	1602
QN	DS	Full	Fine	0.999	0	1985	2308	1986
QN	DS	Local	Coarse	1.000	0	1669	1976	1670
QN	DS	Local	Coarse	0.999	0	1616	1955	1617
QN	DS	Local	Fine	1.000	0	1614	1938	1615
QN	DS	Local	Fine	0.999	0	1671	1938	1672
QN	DS	Mless	Coarse	1.000	0	1553	1983	1555
QN	DS	Mless	Coarse	0.999	0	1716	2155	1719
QN	DS	Mless	Fine	1.000	0	1629	2020	1630
QN	DS	Mless	Fine	0.999	0	1563	1904	1566
DYHS	HZ	LBFGS	—	1.000	0	1227	2455	1228
DYHS	HZ	Full	Coarse	1.000	0	1503	3032	1556
DYHS	HZ	Full	Coarse	0.999	0	1832	3679	1880
DYHS	HZ	Full	Fine	1.000	0	1435	2873	1453
DYHS	HZ	Full	Fine	0.999	0	1825	3655	1845
DYHS	HZ	Local	Coarse	1.000	0	1811	3637	1857
DYHS	HZ	Local	Coarse	0.999	0	1496	3008	1544
DYHS	HZ	Local	Fine	1.000	0	1203	2415	1237
DYHS	HZ	Local	Fine	0.999	0	1132	2269	1156
DYHS	HZ	Mless	Coarse	1.000	0	1701	3443	1797
DYHS	HZ	Mless	Coarse	0.999	0	1662	3393	1825
DYHS	HZ	Mless	Fine	1.000	0	1310	2656	1394
DYHS	HZ	Mless	Fine	0.999	0	1518	3121	1721

β	LS	select	order	τ	status	nit	nf	ng
DYHS	DS	LBFGS	—	1.000	0	1591	3184	1592
DYHS	DS	Full	Coarse	1.000	0	1737	3877	1738
DYHS	DS	Full	Coarse	0.999	0	1991	4362	1992
DYHS	DS	Full	Fine	1.000	0	1932	4142	1933
DYHS	DS	Full	Fine	0.999	0	1958	4248	1959
DYHS	DS	Local	Coarse	1.000	0	1774	3830	1775
DYHS	DS	Local	Coarse	0.999	0	1737	3790	1738
DYHS	DS	Local	Fine	1.000	0	1722	3707	1723
DYHS	DS	Local	Fine	0.999	0	1470	3182	1471
DYHS	DS	Mless	Coarse	1.000	0	1733	4067	1739
DYHS	DS	Mless	Coarse	0.999	0	1793	4135	1805
DYHS	DS	Mless	Fine	1.000	0	1769	4002	1774
DYHS	DS	Mless	Fine	0.999	0	1716	3897	1729
HZ	HZ	LBFGS	—	1.000	0	1337	2676	1339
HZ	HZ	Full	Coarse	1.000	0	1620	3269	1682
HZ	HZ	Full	Coarse	0.999	0	1702	3413	1732
HZ	HZ	Full	Fine	1.000	0	1680	3372	1714
HZ	HZ	Full	Fine	0.999	0	1518	3045	1544
HZ	HZ	Local	Coarse	1.000	0	1357	2736	1415
HZ	HZ	Local	Coarse	0.999	0	1510	3035	1557
HZ	HZ	Local	Fine	1.000	0	1224	2462	1260
HZ	HZ	Local	Fine	0.999	0	1219	2452	1256
HZ	HZ	Mless	Coarse	1.000	0	1638	3307	1716
HZ	HZ	Mless	Coarse	0.999	0	1601	3249	1713
HZ	HZ	Mless	Fine	1.000	0	1361	2742	1419
HZ	HZ	Mless	Fine	0.999	0	1443	2949	1601
HZ	DS	LBFGS	—	1.000	0	1528	3123	1531
HZ	DS	Full	Coarse	1.000	0	1687	3737	1688
HZ	DS	Full	Coarse	0.999	0	1896	4163	1897
HZ	DS	Full	Fine	1.000	0	1827	3977	1829
HZ	DS	Full	Fine	0.999	0	2088	4536	2091
HZ	DS	Local	Coarse	1.000	0	1741	3755	1742
HZ	DS	Local	Coarse	0.999	0	1581	3480	1584
HZ	DS	Local	Fine	1.000	0	1783	3838	1792
HZ	DS	Local	Fine	0.999	0	1587	3419	1595
HZ	DS	Mless	Coarse	1.000	0	2062	4641	2063
HZ	DS	Mless	Coarse	0.999	0	2406	5345	2425
HZ	DS	Mless	Fine	1.000	0	2117	4700	2130
HZ	DS	Mless	Fine	0.999	0	2630	5719	2658

Table 9: Results for problem MINS-DMSA (level 8, 261121 variables)

β	LS	select	order	τ	status	nit	nf	ng
QN	HZ	LBFGS	—	1.000	0	6515	13022	19537
QN	HZ	Full	Coarse	1.000	0	4759	5933	10692
QN	HZ	Full	Coarse	0.999	0	6171	8649	14820
QN	HZ	Full	Fine	1.000	8	8314	10640	18954
QN	HZ	Full	Fine	0.999	0	10751	15170	25921
QN	HZ	Local	Coarse	1.000	0	3926	5190	9116
QN	HZ	Local	Coarse	0.999	0	1380	2189	3569
QN	HZ	Local	Fine	1.000	8	2663	3566	6229
QN	HZ	Local	Fine	0.999	0	2740	4498	7238
QN	HZ	Mless	Coarse	1.000	0	1507	2093	3600
QN	HZ	Mless	Coarse	0.999	0	7496	14260	21756
QN	HZ	Mless	Fine	1.000	0	3440	4790	8230
QN	HZ	Mless	Fine	0.999	0	6746	12845	19591
QN	DS	LBFGS	—	1.000	0	7142	7309	7143
QN	DS	Full	Coarse	1.000	0	2648	4200	2649
QN	DS	Full	Coarse	0.999	0	4756	6436	4757
QN	DS	Full	Fine	1.000	0	7494	11150	7496
QN	DS	Full	Fine	0.999	0	10773	14305	10774
QN	DS	Local	Coarse	1.000	0	3295	4663	3296
QN	DS	Local	Coarse	0.999	0	1487	1835	1488
QN	DS	Local	Fine	1.000	0	4653	6235	4654
QN	DS	Local	Fine	0.999	0	3349	3990	3350
QN	DS	Mless	Coarse	1.000	0	1757	2592	1761
QN	DS	Mless	Coarse	0.999	0	2566	3012	2580
QN	DS	Mless	Fine	1.000	0	2453	3345	2454
QN	DS	Mless	Fine	0.999	0	2275	2614	2282
DYHS	HZ	LBFGS	—	1.000	0	6471	12943	6472
DYHS	HZ	Full	Coarse	1.000	8	976	2162	1315
DYHS	HZ	Full	Coarse	0.999	0	5707	11612	6358
DYHS	HZ	Full	Fine	1.000	8	3550	7326	4260
DYHS	HZ	Full	Fine	0.999	0	8763	17715	9627
DYHS	HZ	Local	Coarse	1.000	8	714	1628	998
DYHS	HZ	Local	Coarse	0.999	0	1868	3807	2072
DYHS	HZ	Local	Fine	1.000	0	5552	11206	6077
DYHS	HZ	Local	Fine	0.999	0	2931	5897	3099
DYHS	HZ	Mless	Coarse	1.000	8	1360	2860	1617
DYHS	HZ	Mless	Coarse	0.999	0	11206	22426	11545
DYHS	HZ	Mless	Fine	1.000	0	2514	5069	2740
DYHS	HZ	Mless	Fine	0.999	0	6592	13218	6856

β	LS	select	order	τ	status	nit	nf	ng
DYHS	DS	LBFGS	—	1.000	0	7822	15664	7823
DYHS	DS	Full	Coarse	1.000	0	3115	8073	3116
DYHS	DS	Full	Coarse	0.999	0	4797	11355	4798
DYHS	DS	Full	Fine	1.000	0	8206	20663	8207
DYHS	DS	Full	Fine	0.999	0	9501	22249	9502
DYHS	DS	Local	Coarse	1.000	0	2836	6831	2837
DYHS	DS	Local	Coarse	0.999	0	1781	3988	1782
DYHS	DS	Local	Fine	1.000	0	5126	11995	5127
DYHS	DS	Local	Fine	0.999	0	1737	3773	1738
DYHS	DS	Mless	Coarse	1.000	0	2537	6241	2542
DYHS	DS	Mless	Coarse	0.999	0	2970	6344	2982
DYHS	DS	Mless	Fine	1.000	0	3472	8311	3474
DYHS	DS	Mless	Fine	0.999	0	3166	6814	3174
HZ	HZ	LBFGS	—	1.000	0	6462	12925	6463
HZ	HZ	Full	Coarse	1.000	8	499	1156	724
HZ	HZ	Full	Coarse	0.999	8	5015	10234	5599
HZ	HZ	Full	Fine	1.000	8	1561	3375	2003
HZ	HZ	Full	Fine	0.999	0	7390	14953	8126
HZ	HZ	Local	Coarse	1.000	0	3290	6724	3775
HZ	HZ	Local	Coarse	0.999	0	1918	3859	2066
HZ	HZ	Local	Fine	1.000	0	4922	9944	5393
HZ	HZ	Local	Fine	0.999	0	2461	4965	2617
HZ	HZ	Mless	Coarse	1.000	8	76	213	143
HZ	HZ	Mless	Coarse	0.999	0	6355	12745	6601
HZ	HZ	Mless	Fine	1.000	0	3092	6257	3370
HZ	HZ	Mless	Fine	0.999	0	6561	13127	6765
HZ	DS	LBFGS	—	1.000	0	6639	13635	6651
HZ	DS	Full	Coarse	1.000	0	3807	9737	3808
HZ	DS	Full	Coarse	0.999	0	5234	11883	5239
HZ	DS	Full	Fine	1.000	0	7318	17302	7320
HZ	DS	Full	Fine	0.999	0	12040	27087	12054
HZ	DS	Local	Coarse	1.000	0	3266	7798	3267
HZ	DS	Local	Coarse	0.999	0	1963	4269	1971
HZ	DS	Local	Fine	1.000	0	3865	8672	3871
HZ	DS	Local	Fine	0.999	0	2126	4574	2129
HZ	DS	Mless	Coarse	1.000	0	2028	4796	2033
HZ	DS	Mless	Coarse	0.999	0	6571	13624	6644
HZ	DS	Mless	Fine	1.000	0	4065	8900	4079
HZ	DS	Mless	Fine	0.999	0	2931	6297	2991

Table 10: Results for problem IGNISC (level 8, 261121 variables)

β	LS	select	order	τ	status	nit	nf	ng
QN	HZ	LBFGS	—	1.000	0	1051	2101	3152
QN	HZ	Full	Coarse	1.000	0	268	420	688
QN	HZ	Full	Coarse	0.999	0	247	376	623
QN	HZ	Full	Fine	1.000	0	198	314	512
QN	HZ	Full	Fine	0.999	0	191	296	487
QN	HZ	Local	Coarse	1.000	0	282	442	724
QN	HZ	Local	Coarse	0.999	0	268	408	676
QN	HZ	Local	Fine	1.000	0	241	378	619
QN	HZ	Local	Fine	0.999	0	214	351	565
QN	HZ	Mless	Coarse	1.000	0	279	421	700
QN	HZ	Mless	Coarse	0.999	0	430	741	1171
QN	HZ	Mless	Fine	1.000	0	185	280	465
QN	HZ	Mless	Fine	0.999	0	143	213	356
QN	DS	LBFGS	—	1.000	0	871	894	872
QN	DS	Full	Coarse	1.000	0	210	276	211
QN	DS	Full	Coarse	0.999	0	301	365	302
QN	DS	Full	Fine	1.000	0	209	250	210
QN	DS	Full	Fine	0.999	0	206	245	207
QN	DS	Local	Coarse	1.000	0	240	299	241
QN	DS	Local	Coarse	0.999	0	192	245	193
QN	DS	Local	Fine	1.000	-7	6	30	25
QN	DS	Local	Fine	0.999	-7	6	31	26
QN	DS	Mless	Coarse	1.000	0	141	185	142
QN	DS	Mless	Coarse	0.999	0	156	202	157
QN	DS	Mless	Fine	1.000	0	157	205	158
QN	DS	Mless	Fine	0.999	0	197	245	199
DYHS	HZ	LBFGS	—	1.000	0	697	1395	698
DYHS	HZ	Full	Coarse	1.000	0	304	609	315
DYHS	HZ	Full	Coarse	0.999	0	224	449	228
DYHS	HZ	Full	Fine	1.000	0	244	490	249
DYHS	HZ	Full	Fine	0.999	0	280	561	288
DYHS	HZ	Local	Coarse	1.000	0	243	487	247
DYHS	HZ	Local	Coarse	0.999	0	234	469	242
DYHS	HZ	Local	Fine	1.000	0	259	519	266
DYHS	HZ	Local	Fine	0.999	0	225	451	230
DYHS	HZ	Mless	Coarse	1.000	0	228	457	237
DYHS	HZ	Mless	Coarse	0.999	0	256	513	283
DYHS	HZ	Mless	Fine	1.000	0	158	317	168
DYHS	HZ	Mless	Fine	0.999	0	182	365	208

β	LS	select	order	τ	status	nit	nf	ng
DYHS	DS	LBFGS	—	1.000	0	1021	2044	1022
DYHS	DS	Full	Coarse	1.000	0	285	630	286
DYHS	DS	Full	Coarse	0.999	0	256	562	257
DYHS	DS	Full	Fine	1.000	0	228	512	229
DYHS	DS	Full	Fine	0.999	0	208	452	209
DYHS	DS	Local	Coarse	1.000	0	308	695	309
DYHS	DS	Local	Coarse	0.999	0	269	609	270
DYHS	DS	Local	Fine	1.000	-7	6	36	25
DYHS	DS	Local	Fine	0.999	-7	6	37	26
DYHS	DS	Mless	Coarse	1.000	0	206	471	207
DYHS	DS	Mless	Coarse	0.999	0	233	526	234
DYHS	DS	Mless	Fine	1.000	0	190	429	191
DYHS	DS	Mless	Fine	0.999	0	340	756	344
HZ	HZ	LBFGS	—	1.000	0	697	1395	698
HZ	HZ	Full	Coarse	1.000	0	249	499	253
HZ	HZ	Full	Coarse	0.999	0	208	417	212
HZ	HZ	Full	Fine	1.000	0	268	537	271
HZ	HZ	Full	Fine	0.999	0	232	466	238
HZ	HZ	Local	Coarse	1.000	0	311	623	317
HZ	HZ	Local	Coarse	0.999	0	218	437	225
HZ	HZ	Local	Fine	1.000	0	239	479	248
HZ	HZ	Local	Fine	0.999	0	197	395	201
HZ	HZ	Mless	Coarse	1.000	0	221	443	227
HZ	HZ	Mless	Coarse	0.999	0	262	525	281
HZ	HZ	Mless	Fine	1.000	0	163	327	174
HZ	HZ	Mless	Fine	0.999	0	134	269	157
HZ	DS	LBFGS	—	1.000	0	1376	2839	1384
HZ	DS	Full	Coarse	1.000	0	195	434	196
HZ	DS	Full	Coarse	0.999	0	325	723	326
HZ	DS	Full	Fine	1.000	-7	66	169	85
HZ	DS	Full	Fine	0.999	0	220	476	221
HZ	DS	Local	Coarse	1.000	0	281	615	282
HZ	DS	Local	Coarse	0.999	0	213	480	214
HZ	DS	Local	Fine	1.000	-7	6	36	25
HZ	DS	Local	Fine	0.999	-7	6	36	25
HZ	DS	Mless	Coarse	1.000	0	232	509	233
HZ	DS	Mless	Coarse	0.999	0	242	531	246
HZ	DS	Mless	Fine	1.000	0	272	604	274
HZ	DS	Mless	Fine	0.999	0	526	1101	533

Table 11: Results for problem DSSC (level 8, 261121 variables)

β	LS	select	order	τ	status	nit	nf	ng
QN	HZ	LBFGS	—	1.000	0	1361	2696	4057
QN	HZ	Full	Coarse	1.000	0	325	494	819
QN	HZ	Full	Coarse	0.999	0	293	448	741
QN	HZ	Full	Fine	1.000	0	282	439	721
QN	HZ	Full	Fine	0.999	0	289	451	740
QN	HZ	Local	Coarse	1.000	0	366	570	936
QN	HZ	Local	Coarse	0.999	0	281	432	713
QN	HZ	Local	Fine	1.000	0	290	464	754
QN	HZ	Local	Fine	0.999	0	307	485	792
QN	HZ	Mless	Coarse	1.000	0	239	350	589
QN	HZ	Mless	Coarse	0.999	0	184	266	450
QN	HZ	Mless	Fine	1.000	0	181	267	448
QN	HZ	Mless	Fine	0.999	0	285	466	751
QN	DS	LBFGS	—	1.000	0	1049	1071	1050
QN	DS	Full	Coarse	1.000	0	221	281	222
QN	DS	Full	Coarse	0.999	0	259	322	260
QN	DS	Full	Fine	1.000	0	254	306	255
QN	DS	Full	Fine	0.999	0	302	354	303
QN	DS	Local	Coarse	1.000	0	232	291	233
QN	DS	Local	Coarse	0.999	0	243	295	244
QN	DS	Local	Fine	1.000	0	292	354	293
QN	DS	Local	Fine	0.999	0	252	308	253
QN	DS	Mless	Coarse	1.000	0	259	336	260
QN	DS	Mless	Coarse	0.999	0	223	282	224
QN	DS	Mless	Fine	1.000	0	167	215	168
QN	DS	Mless	Fine	0.999	0	176	225	179
DYHS	HZ	LBFGS	—	1.000	0	896	1793	897
DYHS	HZ	Full	Coarse	1.000	0	279	559	281
DYHS	HZ	Full	Coarse	0.999	0	214	429	219
DYHS	HZ	Full	Fine	1.000	0	226	453	231
DYHS	HZ	Full	Fine	0.999	0	259	519	261
DYHS	HZ	Local	Coarse	1.000	0	330	661	336
DYHS	HZ	Local	Coarse	0.999	0	304	609	312
DYHS	HZ	Local	Fine	1.000	0	313	627	319
DYHS	HZ	Local	Fine	0.999	0	280	561	292
DYHS	HZ	Mless	Coarse	1.000	0	235	471	246
DYHS	HZ	Mless	Coarse	0.999	0	191	383	206
DYHS	HZ	Mless	Fine	1.000	0	192	385	202
DYHS	HZ	Mless	Fine	0.999	0	193	387	228

β	LS	select	order	τ	status	nit	nf	ng
DYHS	DS	LBFGS	—	1.000	0	1007	2016	1008
DYHS	DS	Full	Coarse	1.000	0	282	630	283
DYHS	DS	Full	Coarse	0.999	0	263	584	264
DYHS	DS	Full	Fine	1.000	0	210	460	211
DYHS	DS	Full	Fine	0.999	0	279	607	280
DYHS	DS	Local	Coarse	1.000	0	324	715	325
DYHS	DS	Local	Coarse	0.999	0	317	692	318
DYHS	DS	Local	Fine	1.000	0	285	621	286
DYHS	DS	Local	Fine	0.999	0	289	617	291
DYHS	DS	Mless	Coarse	1.000	0	261	592	263
DYHS	DS	Mless	Coarse	0.999	0	246	571	250
DYHS	DS	Mless	Fine	1.000	0	153	359	154
DYHS	DS	Mless	Fine	0.999	0	191	434	194
HZ	HZ	LBFGS	—	1.000	0	896	1793	897
HZ	HZ	Full	Coarse	1.000	0	300	624	326
HZ	HZ	Full	Coarse	0.999	0	272	545	274
HZ	HZ	Full	Fine	1.000	0	240	481	245
HZ	HZ	Full	Fine	0.999	0	272	545	277
HZ	HZ	Local	Coarse	1.000	0	318	637	326
HZ	HZ	Local	Coarse	0.999	0	304	609	313
HZ	HZ	Local	Fine	1.000	0	244	489	253
HZ	HZ	Local	Fine	0.999	0	320	641	328
HZ	HZ	Mless	Coarse	1.000	0	248	497	253
HZ	HZ	Mless	Coarse	0.999	0	221	443	236
HZ	HZ	Mless	Fine	1.000	0	227	455	274
HZ	HZ	Mless	Fine	0.999	0	214	429	247
HZ	DS	LBFGS	—	1.000	0	1050	2165	1052
HZ	DS	Full	Coarse	1.000	0	291	655	292
HZ	DS	Full	Coarse	0.999	0	434	944	435
HZ	DS	Full	Fine	1.000	0	281	615	282
HZ	DS	Full	Fine	0.999	0	264	585	265
HZ	DS	Local	Coarse	1.000	0	249	552	250
HZ	DS	Local	Coarse	0.999	0	263	596	264
HZ	DS	Local	Fine	1.000	0	214	473	218
HZ	DS	Local	Fine	0.999	0	501	1046	504
HZ	DS	Mless	Coarse	1.000	0	264	599	265
HZ	DS	Mless	Coarse	0.999	0	4257	8560	4263
HZ	DS	Mless	Fine	1.000	0	326	716	327
HZ	DS	Mless	Fine	0.999	0	338	729	341

Table 12: Results for problem BRATU (level 8, 261121 variables)

β	LS	select	order	τ	status	nit	nf	ng
QN	HZ	LBFGS	—	1.000	0	28933	57848	86781
QN	HZ	Full	Coarse	1.000	0	30063	34683	64746
QN	HZ	Full	Coarse	0.999	0	20870	26531	47401
QN	HZ	Full	Fine	1.000	0	22759	25785	48544
QN	HZ	Full	Fine	0.999	0	21467	28736	50203
QN	HZ	Local	Coarse	1.000	0	11594	13580	25174
QN	HZ	Local	Coarse	0.999	0	40262	79306	119568
QN	HZ	Local	Fine	1.000	0	11681	14198	25879
QN	HZ	Local	Fine	0.999	0	65863	130588	196451
QN	HZ	Mless	Coarse	1.000	0	14422	18488	32910
QN	HZ	Mless	Coarse	0.999	0	460429	919338	344941
QN	HZ	Mless	Fine	1.000	0	15060	19934	34994
QN	HZ	Mless	Fine	0.999	0	423963	846496	317614
QN	DS	LBFGS	—	1.000	12	130051	133142	130051
QN	DS	Full	Coarse	1.000	0	42006	67802	42007
QN	DS	Full	Coarse	0.999	0	24142	34132	24143
QN	DS	Full	Fine	1.000	0	35948	57713	35949
QN	DS	Full	Fine	0.999	12	21063	29400	21063
QN	DS	Local	Coarse	1.000	0	9776	14318	9777
QN	DS	Local	Coarse	0.999	0	5697	6790	5698
QN	DS	Local	Fine	1.000	0	10629	14662	10630
QN	DS	Local	Fine	0.999	0	5718	6707	5719
QN	DS	Mless	Coarse	1.000	0	13101	19192	13109
QN	DS	Mless	Coarse	0.999	0	47200	52292	47363
QN	DS	Mless	Fine	1.000	0	16718	22928	16726
QN	DS	Mless	Fine	0.999	0	19743	22219	19776
DYHS	HZ	LBFGS	—	1.000	0	20449	40677	20674
DYHS	HZ	Full	Coarse	1.000	0	33209	67698	40262
DYHS	HZ	Full	Coarse	0.999	0	20271	40715	23393
DYHS	HZ	Full	Fine	1.000	0	29485	60738	37379
DYHS	HZ	Full	Fine	0.999	0	25171	50532	29772
DYHS	HZ	Local	Coarse	1.000	0	12777	26110	15336
DYHS	HZ	Local	Coarse	0.999	0	123395	246671	123782
DYHS	HZ	Local	Fine	1.000	0	11639	23420	13274
DYHS	HZ	Local	Fine	0.999	0	69692	139206	70166
DYHS	HZ	Mless	Coarse	1.000	0	13272	26827	15141
DYHS	HZ	Mless	Coarse	0.999	0	444361	885886	447771
DYHS	HZ	Mless	Fine	1.000	0	13754	27654	15212
DYHS	HZ	Mless	Fine	0.999	0	439886	878698	441514

β	LS	select	order	τ	status	nit	nf	ng
DYHS	DS	LBFGS	—	1.000	12	130051	260335	130051
DYHS	DS	Full	Coarse	1.000	0	47969	125882	47971
DYHS	DS	Full	Coarse	0.999	0	36051	75684	36052
DYHS	DS	Full	Fine	1.000	0	27314	71534	27315
DYHS	DS	Full	Fine	0.999	0	56895	119970	56896
DYHS	DS	Local	Coarse	1.000	0	10559	25966	10560
DYHS	DS	Local	Coarse	0.999	0	6592	14056	6594
DYHS	DS	Local	Fine	1.000	0	13686	32650	13687
DYHS	DS	Local	Fine	0.999	0	9438	19416	9442
DYHS	DS	Mless	Coarse	1.000	0	14025	34687	14029
DYHS	DS	Mless	Coarse	0.999	0	102389	212689	102668
DYHS	DS	Mless	Fine	1.000	0	17422	42036	17425
DYHS	DS	Mless	Fine	0.999	0	66648	138358	66811
HZ	HZ	LBFGS	—	1.000	0	19982	39846	20104
HZ	HZ	Full	Coarse	1.000	0	37048	75623	44422
HZ	HZ	Full	Coarse	0.999	0	19398	38903	22485
HZ	HZ	Full	Fine	1.000	0	26898	55242	33572
HZ	HZ	Full	Fine	0.999	0	134273	267601	139130
HZ	HZ	Local	Coarse	1.000	0	15004	30626	18036
HZ	HZ	Local	Coarse	0.999	0	50374	100251	51092
HZ	HZ	Local	Fine	1.000	0	12148	24542	14002
HZ	HZ	Local	Fine	0.999	0	113034	225892	113464
HZ	HZ	Mless	Coarse	1.000	0	13218	26669	15068
HZ	HZ	Mless	Coarse	0.999	0	434477	868612	435375
HZ	HZ	Mless	Fine	1.000	0	13439	27029	14757
HZ	HZ	Mless	Fine	0.999	0	361991	722935	363519
HZ	DS	LBFGS	—	1.000	12	130051	267255	130307
HZ	DS	Full	Coarse	1.000	0	41454	105927	41457
HZ	DS	Full	Coarse	0.999	0	32023	74727	32047
HZ	DS	Full	Fine	1.000	0	26062	64157	26064
HZ	DS	Full	Fine	0.999	0	22723	52630	22752
HZ	DS	Local	Coarse	1.000	0	10422	25056	10433
HZ	DS	Local	Coarse	0.999	0	10634	22358	10688
HZ	DS	Local	Fine	1.000	0	13238	30213	13255
HZ	DS	Local	Fine	0.999	0	7976	16631	8012
HZ	DS	Mless	Coarse	1.000	0	15578	35038	15602
HZ	DS	Mless	Coarse	0.999	0	40658	84202	41232
HZ	DS	Mless	Fine	1.000	0	22808	51188	22859
HZ	DS	Mless	Fine	0.999	0	30613	63786	31002

Table 13: Results for problem NCCS (level 7, 130050 variables)

β	LS	select	order	τ	status	nit	nf	ng
QN	HZ	LBFGS	—	1.000	0	29619	59220	88839
QN	HZ	Full	Coarse	1.000	0	31022	35849	66871
QN	HZ	Full	Coarse	0.999	0	22076	28678	50754
QN	HZ	Full	Fine	1.000	0	29235	33036	62271
QN	HZ	Full	Fine	0.999	0	20398	25958	46356
QN	HZ	Local	Coarse	1.000	0	13121	15415	28536
QN	HZ	Local	Coarse	0.999	0	61901	122800	184701
QN	HZ	Local	Fine	1.000	0	19041	22983	42024
QN	HZ	Local	Fine	0.999	0	73108	145307	218415
QN	HZ	Mless	Coarse	1.000	0	13547	17301	30848
QN	HZ	Mless	Coarse	0.999	0	439575	877685	329315
QN	HZ	Mless	Fine	1.000	0	14228	18900	33128
QN	HZ	Mless	Fine	0.999	0	421398	841370	315692
QN	DS	LBFGS	—	1.000	12	130051	133157	130051
QN	DS	Full	Coarse	1.000	0	40156	64776	40157
QN	DS	Full	Coarse	0.999	0	24111	34121	24112
QN	DS	Full	Fine	1.000	0	29683	47535	29684
QN	DS	Full	Fine	0.999	0	24529	35038	24530
QN	DS	Local	Coarse	1.000	0	14849	21873	14850
QN	DS	Local	Coarse	0.999	0	5329	6275	5330
QN	DS	Local	Fine	1.000	0	13445	18705	13447
QN	DS	Local	Fine	0.999	0	6113	7252	6114
QN	DS	Mless	Coarse	1.000	0	14804	21648	14814
QN	DS	Mless	Coarse	0.999	0	56213	61958	56409
QN	DS	Mless	Fine	1.000	0	16572	23005	16578
QN	DS	Mless	Fine	0.999	0	25522	28347	25609
DYHS	HZ	LBFGS	—	1.000	0	21286	42451	21409
DYHS	HZ	Full	Coarse	1.000	0	38748	79016	46265
DYHS	HZ	Full	Coarse	0.999	0	22177	44288	25831
DYHS	HZ	Full	Fine	1.000	0	28494	58710	36273
DYHS	HZ	Full	Fine	0.999	0	26542	53582	31424
DYHS	HZ	Local	Coarse	1.000	0	14398	29487	17440
DYHS	HZ	Local	Coarse	0.999	0	44878	89276	45557
DYHS	HZ	Local	Fine	1.000	0	14097	28562	16634
DYHS	HZ	Local	Fine	0.999	0	70959	141846	71245
DYHS	HZ	Mless	Coarse	1.000	0	13707	27649	15720
DYHS	HZ	Mless	Coarse	0.999	0	410980	820258	413276
DYHS	HZ	Mless	Fine	1.000	0	14738	29659	16387
DYHS	HZ	Mless	Fine	0.999	0	431683	862480	433157
DYHS	DS	LBFGS	—	1.000	12	130051	260339	130051
DYHS	DS	Full	Coarse	1.000	0	41178	108083	41179
DYHS	DS	Full	Coarse	0.999	0	25856	62258	25857
DYHS	DS	Full	Fine	1.000	0	32108	83554	32109
DYHS	DS	Full	Fine	0.999	0	29569	65520	29570
DYHS	DS	Local	Coarse	1.000	0	13284	32614	13285
DYHS	DS	Local	Coarse	0.999	0	5982	12913	5983
DYHS	DS	Local	Fine	1.000	0	13994	33510	13995
DYHS	DS	Local	Fine	0.999	0	6372	13603	6373
DYHS	DS	Mless	Coarse	1.000	0	16287	40373	16290
DYHS	DS	Mless	Coarse	0.999	0	112012	232746	112346
DYHS	DS	Mless	Fine	1.000	0	18368	44421	18372
DYHS	DS	Mless	Fine	0.999	0	51987	108306	52128
HZ	HZ	LBFGS	—	1.000	0	21374	42518	21606
HZ	HZ	Full	Coarse	1.000	0	32607	66663	39382
HZ	HZ	Full	Coarse	0.999	0	19242	38429	22873
HZ	HZ	Full	Fine	1.000	0	31597	65133	39892
HZ	HZ	Full	Fine	0.999	0	20454	41148	24340
HZ	HZ	Local	Coarse	1.000	0	13917	28439	16908
HZ	HZ	Local	Coarse	0.999	0	90512	180394	91390
HZ	HZ	Local	Fine	1.000	0	13131	26560	15145
HZ	HZ	Local	Fine	0.999	0	102106	204120	102495
HZ	HZ	Mless	Coarse	1.000	0	12469	25240	14287
HZ	HZ	Mless	Coarse	0.999	0	424130	847645	425300
HZ	HZ	Mless	Fine	1.000	0	14374	28896	16061
HZ	HZ	Mless	Fine	0.999	0	413701	826422	415240
HZ	DS	LBFGS	—	1.000	12	130051	267428	130337
HZ	DS	Full	Coarse	1.000	0	42900	109942	42901
HZ	DS	Full	Coarse	0.999	0	27659	64424	27674
HZ	DS	Full	Fine	1.000	0	32117	79428	32121
HZ	DS	Full	Fine	0.999	0	26712	61945	26743
HZ	DS	Local	Coarse	1.000	0	15361	36424	15378
HZ	DS	Local	Coarse	0.999	0	8225	17247	8279
HZ	DS	Local	Fine	1.000	0	12422	28276	12429
HZ	DS	Local	Fine	0.999	0	9867	20735	9901
HZ	DS	Mless	Coarse	1.000	0	14768	34142	14820
HZ	DS	Mless	Coarse	0.999	0	53127	110173	53910
HZ	DS	Mless	Fine	1.000	0	36372	80519	36474
HZ	DS	Mless	Fine	0.999	0	31030	64622	31528

Table 14: Results for problem NCCO (level 7, 130050 variables)

β	LS	select	order	τ	status	nit	nf	ng
QN	HZ	LBFGS	—	1.000	0	25283	50538	75821
QN	HZ	Full	Coarse	1.000	0	28261	34072	62333
QN	HZ	Full	Coarse	0.999	0	12786	23779	36565
QN	HZ	Full	Fine	1.000	0	12713	15123	27836
QN	HZ	Full	Fine	0.999	0	16839	22622	39461
QN	HZ	Local	Coarse	1.000	0	9692	11962	21654
QN	HZ	Local	Coarse	0.999	0	13773	27479	41252
QN	HZ	Local	Fine	1.000	0	9108	11273	20381
QN	HZ	Local	Fine	0.999	0	22490	44837	67327
QN	HZ	Mless	Coarse	1.000	0	8398	10854	19252
QN	HZ	Mless	Coarse	0.999	0	30066	60050	90116
QN	HZ	Mless	Fine	1.000	0	7081	9457	16538
QN	HZ	Mless	Fine	0.999	0	15482	30910	46392
QN	DS	LBFGS	—	1.000	0	14348	14706	14349
QN	DS	Full	Coarse	1.000	0	10952	17632	10953
QN	DS	Full	Coarse	0.999	0	15984	22030	15985
QN	DS	Full	Fine	1.000	0	16559	25562	16560
QN	DS	Full	Fine	0.999	0	15474	21580	15475
QN	DS	Local	Coarse	1.000	0	7994	12079	7995
QN	DS	Local	Coarse	0.999	0	6386	6898	6390
QN	DS	Local	Fine	1.000	0	7279	10228	7280
QN	DS	Local	Fine	0.999	0	5110	5590	5113
QN	DS	Mless	Coarse	1.000	0	4316	6350	4319
QN	DS	Mless	Coarse	0.999	0	2446	2688	2456
QN	DS	Mless	Fine	1.000	0	6979	9632	6981
QN	DS	Mless	Fine	0.999	0	4017	4435	4028
DYHS	HZ	LBFGS	—	1.000	0	22442	44885	22443
DYHS	HZ	Full	Coarse	1.000	0	16034	33456	19829
DYHS	HZ	Full	Coarse	0.999	0	16755	33810	17430
DYHS	HZ	Full	Fine	1.000	0	17505	36005	21415
DYHS	HZ	Full	Fine	0.999	0	16080	32601	18037
DYHS	HZ	Local	Coarse	1.000	0	7613	16032	9539
DYHS	HZ	Local	Coarse	0.999	0	26900	53828	26966
DYHS	HZ	Local	Fine	1.000	0	10105	20365	11257
DYHS	HZ	Local	Fine	0.999	0	16662	33328	16678
DYHS	HZ	Mless	Coarse	1.000	0	4759	9886	5700
DYHS	HZ	Mless	Coarse	0.999	0	33220	66453	33258
DYHS	HZ	Mless	Fine	1.000	0	7851	15779	8549
DYHS	HZ	Mless	Fine	0.999	0	21431	42884	21471

β	LS	select	order	τ	status	nit	nf	ng
DYHS	DS	LBFGS	—	1.000	0	12367	24756	12368
DYHS	DS	Full	Coarse	1.000	0	17200	45479	17201
DYHS	DS	Full	Coarse	0.999	0	14249	33704	14250
DYHS	DS	Full	Fine	1.000	0	17764	45739	17765
DYHS	DS	Full	Fine	0.999	0	19856	47571	19857
DYHS	DS	Local	Coarse	1.000	0	4608	11636	4609
DYHS	DS	Local	Coarse	0.999	0	3198	6680	3199
DYHS	DS	Local	Fine	1.000	0	9055	21615	9056
DYHS	DS	Local	Fine	0.999	0	3300	6968	3302
DYHS	DS	Mless	Coarse	1.000	0	6605	16550	6606
DYHS	DS	Mless	Coarse	0.999	0	7204	14924	7234
DYHS	DS	Mless	Fine	1.000	0	9260	22333	9264
DYHS	DS	Mless	Fine	0.999	0	7265	15135	7277
HZ	HZ	LBFGS	—	1.000	0	22442	44885	22443
HZ	HZ	Full	Coarse	1.000	0	9776	20243	11937
HZ	HZ	Full	Coarse	0.999	0	12906	25936	13266
HZ	HZ	Full	Fine	1.000	0	15266	31419	18701
HZ	HZ	Full	Fine	0.999	0	12196	24488	12621
HZ	HZ	Local	Coarse	1.000	0	5098	10736	6411
HZ	HZ	Local	Coarse	0.999	0	9410	18821	9430
HZ	HZ	Local	Fine	1.000	0	5760	11708	6537
HZ	HZ	Local	Fine	0.999	0	15309	30622	15332
HZ	HZ	Mless	Coarse	1.000	0	4467	9296	5377
HZ	HZ	Mless	Coarse	0.999	0	18425	36851	18439
HZ	HZ	Mless	Fine	1.000	0	7271	14666	7965
HZ	HZ	Mless	Fine	0.999	0	22601	45203	22619
HZ	DS	LBFGS	—	1.000	0	12417	25534	12441
HZ	DS	Full	Coarse	1.000	0	16401	42098	16402
HZ	DS	Full	Coarse	0.999	0	13789	31642	13800
HZ	DS	Full	Fine	1.000	0	14625	35445	14627
HZ	DS	Full	Fine	0.999	0	12178	27812	12200
HZ	DS	Local	Coarse	1.000	0	4657	11261	4662
HZ	DS	Local	Coarse	0.999	0	3357	7093	3384
HZ	DS	Local	Fine	1.000	0	8977	20563	8988
HZ	DS	Local	Fine	0.999	0	4149	8706	4172
HZ	DS	Mless	Coarse	1.000	0	3276	7639	3280
HZ	DS	Mless	Coarse	0.999	0	4795	9994	4871
HZ	DS	Mless	Fine	1.000	0	10837	23856	10853
HZ	DS	Mless	Fine	0.999	0	5260	11116	5353

Table 15: Results for problem MOREBV (level 8, 261121 variables)